

Counting chiral operators in quiver gauge theories

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ABSTRACT: We discuss in detail the problem of counting BPS gauge invariant operators in the chiral ring of quiver gauge theories living on D-branes probing generic toric CY singularities. The computation of generating functions that include counting of baryonic operators is based on a relation between the baryonic charges in field theory and the Kähler moduli of the CY singularities. A study of the interplay between gauge theory and geometry shows that given geometrical sectors appear more than once in the field theory, leading to a notion of “multiplicities”. We explain in detail how to decompose the generating function for one D-brane into different sectors and how to compute their relevant multiplicities by introducing geometric and anomalous baryonic charges. The Plethystic Exponential remains a major tool for passing from one D-brane to arbitrary number N of D-branes. Explicit formulae are given for few examples, including $\mathbb{C}^3/\mathbb{Z}_3$, \mathbb{F}_0 , and dP_1 .

KEYWORDS: AdS-CFT Correspondence, Gauge-gravity correspondence.

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1. Introduction

Recently, there has been growing interest in “counting” chiral BPS operators in field theories which arise on the world-volume of branes probing Calabi-Yau singularities [1–18]. Determining the matter content and the interactions of these field theories is an interesting and nontrivial question in itself. The study of this problem began with orbifolds [19–22] and much progress has been made in understanding toric singularities as well [23–29]. In the toric case brane tilings, a generalization of the brane boxes [30–32], allow for a great simplification of the problem by providing a very geometric viewpoint [33–41].

The chiral BPS operators in question are dual to D3-branes wrapped on generically nontrivial three-cycles on the gravity side [42]. Branes on trivial cycles are termed (dual) giant gravitons [43–45] and are dual to mesonic operators while branes on nontrivial cycles are dual to baryonic operators [42, 46, 47]. There is a relation between giant gravitons or baryons and holomorphic curves in the Calabi-Yau which was first discussed in [48, 49]. As a consequence, combinatorial data of BPS operators can be packed into generating functions for holomorphic curves [9, 50], which contain ample information about the geometry of the singularity.

This paper is devoted to the study of the (baryonic and mesonic) generating function for the chiral ring in quiver gauge theories. Extending the results of [12], we compute the generating functions including baryonic degrees of freedom for various theories. We first study in detail the generating function for one D-brane and we decompose it into sectors

with definite baryonic charges. This decomposition is closely related to the geometry and to the generating functions for holomorphic curves obtained by localization in the Calabi-Yau manifold. We conjecture that the generating function for a number N of D-branes is completely determined by the generating function for a single D-brane and it is obtained by applying the plethystic exponential to each sector [6, 9–12]. This conjecture, which can be proved in the case of mesonic operators, is inspired by the geometrical quantization of the classical D3-brane configurations in the gravitational dual. We explicitly compute the generating functions for a selected set of singularities, including $\mathbb{C}^3/\mathbb{Z}_3$, \mathbb{F}_0 and dP_1 , and we make various checks in the dual field theory.

In [12] we studied the simpler and elegant cases of the conifold and the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold. A new feature, which arises for more involved singularities, like for example $\mathbb{C}^3/\mathbb{Z}_3$, \mathbb{F}_0 and dP_1 , is the existence of multiplicities, namely the fact that geometrical sectors appear more than once in field theory. As we go over these examples in detail, we find that multiplicities have a geometrical interpretation and can be resolved, with a construction that ties together in a fascinating way the algebraic geometry of the CY and the combinatorics of quiver data.

The paper is organized as follows. In section 2, we discuss the basics of generating functions and plethystics. In section 3, we apply these tools and give a brief review of the conifold example. Section 4 continues with the detailed discussion of the partition functions from both the field theory and the geometry perspectives. The GKZ decomposition is introduced and the auxiliary GKZ partition function is defined. Section 5 contains detailed examples, based on $\mathbb{C}^3/\mathbb{Z}_3$, \mathbb{F}_0 and dP_1 , and explicitly computes generating functions both for $N = 1$ and for small $N > 1$. Section 6 deals with $N > 1$ D-branes and gives a systematic approach to the field theory computation by means of the Molien formula. Appendices A and B contain some preliminary discussion and observations about singular horizons and a discussion of the anomalous baryonic charges from the viewpoint of the dual *shiver* construction.

Finally, a useful list of notations is reported in appendix C.

2. General structure of generating functions for BPS operators

In this section we will give general prescriptions on the computation of generating functions for BPS operators in the chiral ring of a supersymmetric gauge theory that lives on a D-brane which probes a generic non-compact Calabi-Yau manifold. The simpler cases of the conifold and the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold were discussed in detail in [12].

Given an $\mathcal{N} = 1$ supersymmetric gauge theory with a collection of $U(1)$ global symmetries, $\prod_{i=1}^r U(1)_i$, we have a set of r *chemical potentials* $\{t_i\}_{i=1}^r$. The generating function for a gauge theory living on a D-brane probing a generic non-compact CY manifold depends on the set of parameters, t_i . There is always at least one such $U(1)$ global symmetry and one such chemical potential t , corresponding to the $U(1)_R$ symmetry.

The global charges are divided into classes: *baryonic charges*, and *flavor charges* (by abuse of language, we will include the R -symmetry in this latter class). The number of non-anomalous flavor symmetries, related to the isometries of the CY, is less than three while the number of non-anomalous baryonic symmetries, related to the group of divisors in

the CY, can be quite large. In certain cases, we can also have baryonic discrete symmetries. As is demonstrated below, in addition to the non-anomalous baryonic charges we need to consider the *anomalous baryonic charges*. We will only consider the case of toric CY where the number of flavor symmetries is three. When it will be necessary to make distinctions, we will denote with x, y or q_i the flavor chemical potentials and with b_i the non-anomalous baryonic chemical potentials. Chemical potentials for anomalous charges are denoted by a_i .

For a given CY manifold, we denote the generating function for N D-branes by $g_N(\{t_i\}; CY)$. The generating function for $N = 1$ is simple to compute by using field theory arguments. Recall that the quiver gauge theory living on the world-volume of the D3-branes, which can be determined in the toric case using dimer technology [33, 34],¹ consists of a gauge group $SU(N)^G$, adjoint or bi-fundamental chiral fields² \mathbf{X}^J , which can be considered as $N \times N$ matrices, and a superpotential $W(\mathbf{X}^J)$.

For $N = 1$ the matrices \mathbf{X}^J reduce to numbers and the F-term conditions become polynomial equations in the commuting numbers \mathbf{X}^J . We can consider the polynomial ring $\mathbb{C}[\mathbf{X}^J]$ to be graded by the weights t_i . Since the gauge group is acting trivially for $N = 1$, the ring of gauge invariants is just the quotient ring

$$\mathcal{R}_{N=1}^{\text{inv}} = \mathbb{C}[\mathbf{X}^J]/\mathcal{I}$$

where \mathcal{I} is the set of F-term constraints $dW(\mathbf{X}^J)/d\mathbf{X}^J$. The generating function for polynomial rings is called Hilbert series in the mathematical literature and can be computed in an algorithmic way. In particular, computer algebra programs, like Macaulay2 [66], have built-in commands to compute these generating functions. We can therefore assume that the generating function $g_1(\{t_i\})$ for $\mathcal{R}_{N=1}^{\text{inv}}$ is known.

We proceed to the determination of g_N with a general conjecture:

- *For the class of theories considered here (D-branes probing non-compact CY which are any of toric, orbifolds or complete intersections), the knowledge of the generating function for $N = 1$ is enough to compute the generating function for any N .*

This is a familiar fact for mesonic generating functions [6], and it is essentially due to the fact that the operators for finite N are symmetric functions of the operators for $N = 1$. This is also familiar for baryonic generating functions, where the knowledge of a single generating function, $g_{1,B}$, for one D-brane, $N = 1$ and baryon number B , is enough to compute all generating functions for any number of D-branes and for a fixed baryonic number [9].

The general construction is as follows. There exists a decomposition of the $N = 1$ ring of invariants $\mathcal{R}_{N=1}^{\text{inv}}$, and consequently of its generating function, into sectors \mathcal{S} of definite baryonic charges

$$g_1(\{t_i\}; CY) = \sum_{\mathcal{S}} g_{1,\mathcal{S}}(\{t_i\}; CY) \tag{2.1}$$

¹See [36–41, 51–65] for a rich set of subsequent developments.

²Henceforth we denote fields by bold characters to distinguish them from global quantum numbers.

where $g_{1,\mathcal{S}}$ is the generating function for the subsector $\mathcal{S} \subset \mathcal{R}_{N=1}^{\text{inv}}$. All elements in \mathcal{S} have the same baryonic charges, and, except for a multiplicative factor, $g_{1,\mathcal{S}}$ only depends on the flavor charges q_i . In simple cases, like the conifold, \mathcal{S} is just a label running over all the possible values of the non-anomalous baryonic charge. The understanding of the precise decomposition of $\mathcal{R}_{N=1}^{\text{inv}}$ into subsectors in the general case is a nontrivial task and is one of the subjects of this paper.

The generating function for N branes is then obtained by taking N -fold symmetric products of elements in each given sector \mathcal{S} . This is precisely the role which is played by the Plethystic Exponential (PE) — to take a generating function for a set of operators and count all possible symmetric functions of it. If we introduce a chemical potential ν for the number of D-branes, the generating function for any number of D-branes is given by

$$\begin{aligned}
 g(\nu; \{t_i\}; CY) &= \sum_{\mathcal{S}} \text{PE}_{\nu}[g_{1,\mathcal{S}}(\{t_i\}; CY)] \equiv \sum_{\mathcal{S}} \exp\left(\sum_{k=1}^{\infty} \frac{\nu^k}{k} g_{1,\mathcal{S}}(\{t_i^k\}; CY)\right) \\
 &\equiv \sum_{N=0}^{\infty} g_N(\{t_i\}; CY) \nu^N
 \end{aligned}
 \tag{2.2}$$

The detailed description of the decomposition into sectors \mathcal{S} is given in the rest of this paper, but it is important to notice from the very beginning that such a decomposition is not unique. As already mentioned above, gauge invariants in the same sector have the same baryonic charges. One can take these baryonic charges to be non-anomalous. This however, turns out to be not enough and we seem to need a finer decomposition of the ring of invariants which is obtained by enlarging the set of non-anomalous baryonic charges to a larger set. There are two basic extensions, one related to an expansion in a full set of baryonic charges, anomalous and non-anomalous, and the other extension is related to a full set of discretized Kähler moduli on the CY resolutions. We thus have two complementary points of view:

- **Quantum field theory perspective:** the most general decomposition of the generating function $g_1(\{t_i\})$ is into the full set of baryonic charges. Let us extend the set of chemical potentials t_i to all the baryonic charges, including the anomalous ones, denoted by a_i . There are $G - 1$ independent baryonic charges, where G is the number of gauge groups. We can thus decompose $\mathcal{R}_{N=1}^{\text{inv}}$ into sectors with definite charges under $U(1)^{G-1}$. $g_1(\{t_i\})$ will decompose into a formal Laurent series in the baryonic chemical potentials b_i and a_i . The $\mathcal{R}_N^{\text{inv}}$ rings of invariants for number of colors N will similarly decompose into sectors of definite baryonic charge. We can formally extend the gauge group $SU(N)^G$ to $U(N)^G/U(1)$ by gauging the baryonic symmetries.³ From this perspective, the decomposition of the ring of $SU(N)^G$ invariants into Abelian representations of the extended group $U(N)^G/U(1)$ is sometimes called an expansion in *covariants* and is extremely natural from the point of view of invariant theory. All sectors \mathcal{S} appear with multiplicity one in the decomposition of equation (2.2).

³The theory will of course be anomalous. The overall $U(1)$ is discarded since it acts trivially.

- **The dual geometrical perspective:** the full set of BPS states of the dual gauge theory can be obtained by quantizing the classical configuration space of supersymmetric D3-branes wrapped on the horizon. This problem can be equivalently rephrased in terms of holomorphic surfaces in the CY with g_1 as generating function [48, 49]. Quite remarkably, g_1 has a decomposition

$$g_1(\{t_i\}; CY) = \sum_{\mathcal{S}} m(\mathcal{S}) g_{1,\mathcal{S}}(\{t_i\}; CY), \tag{2.3}$$

where the parameters \mathcal{S} can be identified with a complete set of *discretized Kähler moduli* and the integers $m(\mathcal{S})$ are multiplicities. We will call it the *GKZ decomposition*, from the known description of the Kähler cone in terms of a secondary fan given by the GKZ construction (for a useful reference see [67]). The functions $g_{1,\mathcal{S}}$ can be explicitly determined with the computation of a character using the equivariant index theorem. This geometrical decomposition has multiplicities $m(\mathcal{S})$ which will be interpreted in the following sections and discussed in detail in section 4. The result for finite N is generated by the following function

$$g(\nu; \{t_i\}; CY) = \sum_{\mathcal{S}} m(\mathcal{S}) \text{PE}_{\nu}[g_{1,\mathcal{S}}(\{t_i\}; CY)], \tag{2.4}$$

and can be interpreted as the result of quantizing the classical BPS D3-brane configuration in each sector \mathcal{S} .

The two decompositions of the $N = 1$ generating function are different and complementary. For a toric CY manifold that has a toric diagram with d external vertices and I internal integral points, the number of non-anomalous baryonic symmetries is $d - 3$, the number of anomalous baryonic symmetries is $2I$ and the dimension of the Kähler moduli space is $d - 3 + I$. The field theory expansion is thus based on a lattice $\Gamma_{(b,a)}$ of dimension $d - 3 + 2I$ consisting of all baryonic charges, anomalous or not, while the geometrical expansion is based on a lattice Γ_{GKZ} of dimension $d - 3 + I$. The two sets have a nontrivial intersection Γ_b , consisting of non-anomalous baryonic charges.

At the end, we will be interested in the generating function for BPS operators with chemical potential with respect to the *non-anomalous* charges. To this purpose, we must project any of the two lattices on their intersection, which is the $d - 3$ lattice of non-anomalous baryonic symmetries Γ_b

$$g_1(\{t_i\}) = \sum_{k \in \Gamma_b} m(k) g_{1,k}(\{t_i\})$$

and multiplicities will generically appear.

On the other hand, we could even imagine to enlarge our lattices. Adding the anomalous baryonic charges to the GKZ fan we obtain a lattice of dimension $d - 3 + 3I$. The points give hollow polygons over the GKZ fan. All these issues will be discussed in detail in the rest of the paper.

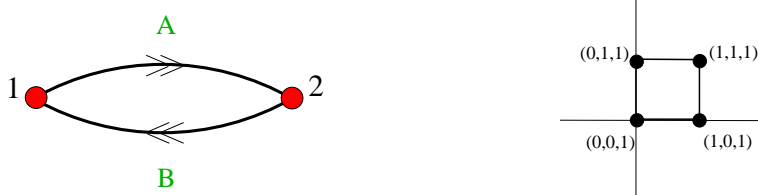


Figure 1: Quiver and toric diagram for the conifold.

	SU(2) ₁		SU(2) ₂		U(1) _R	U(1) _B	monomial
	j_1	m_1	j_2	m_2			
A ₁	$\frac{1}{2}$	$+\frac{1}{2}$	0	0	$\frac{1}{2}$	1	$t_1 x$
A ₂	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	1	$\frac{t_1}{x}$
B ₁	0	0	$\frac{1}{2}$	$+\frac{1}{2}$	$\frac{1}{2}$	-1	$t_2 y$
B ₂	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$\frac{t_2}{y}$

Table 1: Global charges for the basic fields of the quiver gauge theory living on the D-brane probing the conifold.

3. Review of the conifold example

To demonstrate our general discussion above and to prepare for more involved cases we start by reviewing the generating function for the conifold.

The gauge theory on the conifold has a global symmetry $SU(2)_1 \times SU(2)_2 \times U(1)_R \times U(1)_B$. It has four basic fields **A**_{1,2} and **B**_{1,2} that transform under these symmetries according to table 1.

The last column represents the corresponding monomial in the generating function for BPS operators in the chiral ring. t_1 is the chemical potential for the number of **A** fields, t_2 is the chemical potential for the number of **B** fields, x is the chemical potential for the Cartan generator of $SU(2)_1$, and y is the chemical potential for the Cartan generator of $SU(2)_2$.

The theory has a single baryonic charge $U(1)_B$ which is not anomalous. We can introduce a corresponding chemical potential b . With this notation we have $t_1 = tb$ and $t_2 = \frac{t}{b}$. The chemical potentials t and b keep track of the R-charge and the baryonic charge B , respectively.

Since the F-terms of the theory are

$$\mathbf{A}_1 \mathbf{B}_i \mathbf{A}_2 - \mathbf{A}_2 \mathbf{B}_i \mathbf{A}_1 = 0 \quad \mathbf{B}_1 \mathbf{A}_i \mathbf{B}_2 - \mathbf{B}_2 \mathbf{A}_i \mathbf{B}_1 = 0 \quad i = 1, 2$$

they vanish for $N = 1$. The $N = 1$ generating function is thus freely generated by the four basic fields of the conifold gauge theory and it takes the form

$$g_1(t_1, t_2, x, y; \mathcal{C}) = \frac{1}{(1 - t_1 x)(1 - \frac{t_1}{x})(1 - t_2 y)(1 - \frac{t_2}{y})}. \quad (3.1)$$

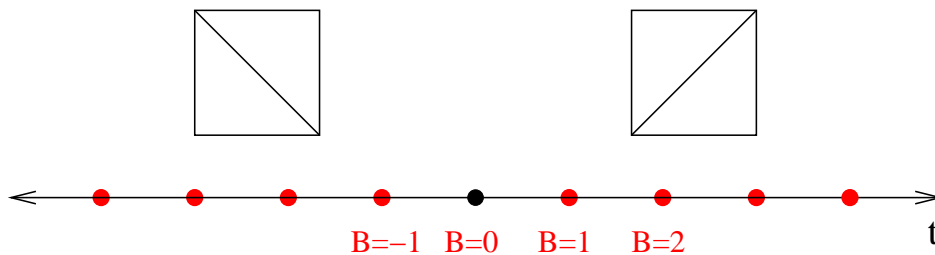


Figure 2: The GKZ decomposition for the Kähler moduli space of the conifold, consisting of two one-dimensional cones connected by a flop. The coordinate t on the moduli space is associated with the volume of the two-cycle in the resolution of the conifold. When t goes to zero, the cycle vanishes and we can perform a flop on the variety by inflating a different two-cycle. A natural discretization of the GKZ fan is in correspondence with the decomposition of the g_1 generating function.

In the following we set $x = y = 1$ for simplicity. General formulae including the $SU(2)$ chemical potentials can be found in [12] and in section 5.

g_1 decomposes into sectors with fixed baryonic charge B , each with multiplicity one:

$$\begin{aligned}
 g_1(t_1, t_2; \mathcal{C}) &= \sum_{B=-\infty}^{\infty} g_{1,B}(t_1, t_2; \mathcal{C}), \\
 g_{1,B>0}(t_1, t_2; \mathcal{C}) &= \sum_{n=0}^{\infty} (n+1+B)(n+1)t_1^{n+B}t_2^n \\
 g_{1,B<0}(t_1, t_2; \mathcal{C}) &= \sum_{n=0}^{\infty} (n+1)(n+1+|B|)t_1^n t_2^{n+|B|}
 \end{aligned} \tag{3.2}$$

It is manifest that each term in $g_{1,B}$ has a monomial b^B corresponding to a baryonic charge B . The decomposition into each baryonic charge can be computed by expanding $g_1(t, b; \mathcal{C})$ in a formal Laurent series in b or by determining the functions $g_{1,B}$ by resolving the CY, see figure 2, and using the equivariant index theorem. Both computations have been discussed in detail in [12] and will be reviewed in the following sections.

The decomposition has a clear interpretation in terms of supersymmetric D3-branes states: $g_{1,B}$ counts the supersymmetric D3-branes wrapping cycles of homology B thus corresponding to states with baryonic number B . Quite remarkably, the structure of the integer lattices with generating function $g_{1,B}$ and the explicit computation with the index theorem, strongly suggests a relation between B and a discretized Kähler modulus of the resolved CY [12].

The result for generic N is obtained as follows

$$\begin{aligned}
 g(\nu; t_1, t_2; \mathcal{C}) &= \sum_{B=-\infty}^{\infty} \text{PE}_{\nu}[g_{1,B}(t_1, t_2; \mathcal{C})], \\
 g(\nu; t_1, t_2; \mathcal{C}) &= \sum_{N=0}^{\infty} \nu^N g_N(t_1, t_2; \mathcal{C})
 \end{aligned} \tag{3.3}$$

Here we list the generating functions for small values of N

$$g_2(t_1, t_2; \mathcal{C}) = \frac{1 + t_1 t_2 + t_1^2 t_2^2 - 3t_1^4 t_2^2 - 3t_1^2 t_2^4 + t_1^5 t_2^3 + t_1^3 t_2^5 - 3t_1^3 t_2^3 + 4t_1^4 t_2^4}{(1 - t_1^2)^3 (1 - t_1 t_2)^3 (1 - t_2^2)^3}. \quad (3.4)$$

$$g_3(t_1, t_2; \mathcal{C}) = \frac{F(t_1, t_2)}{(1 - t_1^3)^4 (1 - t_1 t_2)^3 (1 - t_1^2 t_2^2)^3 (1 - t_2^3)^4}, \quad (3.5)$$

$$\begin{aligned} F(t_1, t_2) = & 1 + t_1^{15} t_2^9 + 3t_1^{14} t_2^8 (-1 + 2t_2^3) + t_1 (t_2 + 2t_2^4) + t_1^2 (7t_2^2 - 4t_2^5) + t_1^3 (7t_2^3 - 10t_2^6) \\ & + 3t_1^{13} t_2^7 (-1 + 2t_2^6) + t_1^{12} t_2^6 (9 - 34t_2^3 + 22t_2^6) + t_1^7 t_2^4 (-22 + 35t_2^3 + 8t_2^6 - 3t_2^9) \\ & - t_1^8 t_2^5 (4 - 35t_2^3 + 10t_2^6 + 3t_2^9) + t_1^6 t_2^3 (-10 - t_2^3 + 4t_2^6 + 9t_2^9) \\ & + t_1^{10} (6t_2^4 + 8t_2^7 - 26t_2^{10}) + t_1^9 t_2^6 (4 + 31t_2^3 - 34t_2^6 + t_2^9) \\ & + 2t_1^{11} t_2^5 (3 - 5t_2^3 - 7t_2^6 + 3t_2^9) + 2t_1^4 (t_2 + t_2^4 - 11t_2^7 + 3t_2^{10}) \\ & + 2t_1^5 t_2^2 (-2 - 5t_2^3 - 2t_2^6 + 3t_2^9) \end{aligned} \quad (3.6)$$

4. Expanding the $N = 1$ generating function

Our decomposition of the ring of invariants of the gauge theory is a decomposition into different types of *determinants*. For simplicity, we will use the following notation: given a pair of gauge groups (α, β) , $\alpha, \beta = 1, \dots, G$, we call *determinant of type* (α, β) a gauge invariant of the form

$$\epsilon^{i_1, \dots, i_N} (\mathbf{X}_{I_1}^{(\alpha, \beta)})_{i_1}^{j_1} \dots (\mathbf{X}_{I_N}^{(\alpha, \beta)})_{i_N}^{j_N} \epsilon_{j_1, \dots, j_N}$$

where $(\mathbf{X}_I^{(\alpha, \beta)})_i^j$ denotes a string of elementary fields with all gauge indices contracted except two indices, i and j , corresponding to the gauge groups (α, β) . The index I runs over all possible strings of elementary fields with these properties. The full set of invariants is obtained by arbitrary products of these determinants. Using the tensor relation

$$\epsilon^{i_1, \dots, i_N} \epsilon_{j_1, \dots, j_N} = \delta_{[j_1}^{i_1} \dots \delta_{j_N]}^{i_N}$$

some of these products of determinants are equivalent and some of these are actually equivalent to mesonic operators made only with traces. In particular, mesons are included in the above description as determinants of type (α, α) .

We can decompose the ring of invariants according to the baryonic charges, which indeed distinguish between different types of determinants, or baryons. This decomposition is natural in field theory and it also has a simple interpretation in the dual gravity theory.

In fact, in theories obtained from D3-branes at CY singularities, baryons can be identified with branes wrapped on nontrivial cycles on the base H of the CY. The non-anomalous symmetries can be clearly identified in the dual theory. In particular, states with the same non-anomalous baryonic charges can be continuously deformed into each other: we can thus relate the set of non-anomalous baryonic charges to the group of three-cycles in H . The homology $H^3(H, \mathbb{Z}) = \mathbb{Z}^{d-3} \times \Gamma$ determines $d - 3$ continuous baryonic charges (d is the number of vertices of the toric diagram) and possibly a set of discrete baryonic charges from the torsion part Γ .

In the case of the conifold there is one non-anomalous baryonic charge (since $d = 4$) which is related to the single three-cycle in the base $T^{1,1}$. There are only two gauge groups

and two types of determinants: (1, 2) and (2, 1). The invariants decompose according to the baryonic charge:

1. $B = 0$ corresponds to the mesons (D3-branes wrapping trivial cycles, a. k. a. giant gravitons [43]),
2. $B > 0$ corresponds to the sector containing the determinants $(\det \mathbf{A})^B$ and all possible mesonic excitations (D3-branes wrapping B times the 3-cycle),
3. finally, $B < 0$ corresponds to the sector containing the determinants $(\det \mathbf{B})^{|B|}$ and all possible mesonic excitations (D3-branes wrapping $|B|$ times the 3-cycle with opposite orientation).

The conifold picture is nice and in many ways elegant. However, a simple look at any other quiver gauge theory reveals that this simple picture is too naive. Consider, for example, the case of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$ (see figure 4), that already reveals all types of oddities:

- Since $d = 3$, there is no continuous non-anomalous baryonic symmetry. However, $H^3(S^5/\mathbb{Z}_3, \mathbb{Z}) = \mathbb{Z}_3$ and there is a discrete baryonic symmetry. We can indeed construct determinants, for example, using the fields \mathbf{U}, \mathbf{V} and \mathbf{W} with \mathbb{Z}_3 charge +1. These do not carry any continuous conserved charge since the product $\det \mathbf{U} \det \mathbf{V} \det \mathbf{W}$ can be rewritten as a meson in terms of traces; for example, using $\epsilon^{i_1, \dots, i_N} \epsilon_{j_1, \dots, j_N} = \delta_{[j_1}^{i_1} \dots \delta_{j_N]}^{i_N}$ we can write,

$$\det \mathbf{U}_1 \det \mathbf{V}_1 \det \mathbf{W}_1 = \det(\mathbf{U}_1 \mathbf{V}_1 \mathbf{W}_1) = \text{Tr}(\mathbf{U}_1 \mathbf{V}_1 \mathbf{W}_1)^N + \dots \pm (\text{Tr} \mathbf{U}_1 \mathbf{V}_1 \mathbf{W}_1)^N$$

On the other hand, $(\det \mathbf{U}_1)^3$ cannot be reduced to traces simply because there are no gauge invariant traces we can make with \mathbf{U}_1 alone: we have actually an infinite number of products of determinants ($(\det \mathbf{U})^n$ for $n = 1, 2, \dots$ for example) that cannot be rewritten in terms of mesons. All these operators correspond in the ring of invariants to sectors that cannot be distinguished by the discrete baryonic charge.

- The BPS D3-brane configurations wrap divisors in the CY: for $\mathbb{C}^3/\mathbb{Z}_3$ we have just a single divisor D satisfying $3D = 0$ and this agrees with the homology of the base S^5/\mathbb{Z}_3 . However, we also have a vanishing compact four-cycle which is represented in toric geometry by the integer internal point of the toric diagram. The size of this cycle becomes finite when we blow up the orbifold. It is conceivable that the inclusion of compact four-cycles such as this one will affect the description of the classical configuration space of D3-branes. This will add a new parameter, related to the group of divisors on the CY resolution, which has dimension one.
- We could distinguish among elementary fields and types of determinants by using all the possible baryonic charges, including the anomalous ones. For $\mathbb{C}^3/\mathbb{Z}_3$ this would lead to the inclusion of the two existing anomalous baryonic charges. As we will explain, this set of charges is different and complementary with respect to that related to the group of divisors on the resolution.

Having this example in mind, we now discuss the two possible expansions of the ring of invariants. Explicit examples of the decompositions will be presented in section 5. We encourage the reader to jump forth and back with section 5 for a better understanding of the material.

4.1 Expanding in a complete set of baryonic charges

The most general decomposition of the $g_1(\{t_i\})$ generating function is according to the full set of baryonic charges, including the anomalous ones, denoted by a_i . The sectors \mathcal{S} in this case correspond to sectors with definite anomalous and non-anomalous baryonic charges.

There are $G - 1$ independent baryonic charges, where G is the number of gauge groups. By gauging the baryonic symmetries, we would obtain a quiver theory with the same fields and superpotential, and a gauge group

$$\prod_{i=1}^G \text{U}(N) / \text{U}(1),$$

where we factor out the overall decoupled $\text{U}(1)$. Some of the $\text{U}(1)$ factors will be anomalous, of course. The baryonic charges have a very natural description: they correspond to the $\text{U}(1)$ factors in $\text{U}(N)^G / \text{U}(1)$. In this way, different elementary fields have the same baryonic charges if and only if they are charged under the same gauge groups. This allows to efficiently distinguish between invariants belonging to different sectors. Notice that non-anomalous baryonic symmetries alone would not distinguish all inequivalent possibilities. For example, in $\mathbb{C}^3 / \mathbb{Z}_3$, the mesonic operator $\det \mathbf{U}_1 \det \mathbf{V}_1 \det \mathbf{W}_1$ and the determinant $(\det \mathbf{U}_1)^3$ have the same charge under \mathbb{Z}_3 , but different charges under the two anomalous baryonic symmetries.

Let us thus extend the set of chemical potentials t_i to all the baryonic charges, including the anomalous ones. We can therefore decompose $\mathcal{R}_{N=1}^{\text{inv}}$ into sectors with definite charges under $\text{U}(1)^{G-1}$.

The $N = 1$ generating function $g_1(\{t_i\})$ will decompose into a formal Laurent series in the baryonic chemical potentials b_i and a_i . The explicit decomposition of g_1 into a formal Laurent series can be done by repeatedly applying the residue theorem; the computation however quickly becomes involved, since the order of integration becomes crucial and divides the result into many different cases. The ring of invariants $\mathcal{R}_N^{\text{inv}}$, will similarly decompose into sectors of definite baryonic charges. The generating function $g_N(\{t_i\})$ can then be computed according to equation (2.2).

We can understand this decomposition in terms of representation theory. From this perspective, we have decomposed the ring of $\text{SU}(N)^G$ invariants into Abelian representations of the extended group $\text{U}(N)^G / \text{U}(1)$. This is sometimes called an expansion in *covariants* and is extremely natural from the point of view of invariant theory. From our point of view, covariants are just the possible set of independent *determinants*. Each sector \mathcal{S} in $\mathcal{R}_N^{\text{inv}}$ will be specified by a certain number of gauge group pairs (α_i, β_i) and is associated to the subsector of the ring of gauge invariants made with products of the *determinants* of type (α_i, β_i) .

To make connection with the toric quiver gauge theories we notice that we have the relation [29]:

$$G - 1 = 2I + (d - 3) \tag{4.1}$$

where d is the number of vertices of the toric diagram and I the number of integer internal points. Only $d - 3$ of these baryonic symmetries are not anomalous.

We expect all sectors \mathcal{S} to appear with multiplicity one in the decomposition of equation (2.2).

4.2 Supersymmetric D3-branes and the GKZ decomposition

The full set of BPS states of the dual gauge theory can be obtained by quantizing the classical configuration space of supersymmetric D3-branes wrapped on the CY horizon [48, 49, 4]. The supersymmetric D3-brane configurations are in one-to-one correspondence with holomorphic four-cycles in the CY, or divisors [48]. This is clear for static D3-branes wrapping a three-cycle in the horizon: the corresponding divisor is the cone over the three-cycle. For more general configurations of excited and rotating D3-branes we obtain a four-cycle by a Euclidean continuation: we can replace time with the radial coordinate using the isometries of (Euclidean) AdS_5 . Our problem can be equivalently rephrased in terms of holomorphic surfaces in the CY manifold with g_1 as a generating function.

From this perspective, we have an obvious decomposition into sectors \mathcal{S} corresponding to Euclidean D3-branes that can be continuously deformed into each other in the CY. Such D3-branes have the same non-anomalous baryonic numbers; indeed, geometrically, the non-anomalous baryonic charges are identified with the group of divisors modulo linear equivalence. Let us discuss this point in detail for toric varieties since it will be crucial in the following.

Recall that our conical CY is specified by a toric diagram in the plane with d vertices having integer coordinates n_i . By embedding the plane in three dimensions we have a toric cone with edges $V_i = (n_i, 1) \in \mathbb{Z}^3$ (the toric fan of our conical CY). Call the set of edges $\Sigma(1)$. Assign a ‘‘homogeneous coordinate’’ x_i to each $V_i \in \Sigma(1)$; the x_i span \mathbb{C}^d . Consider the group

$$K = \left\{ (\mu_1, \dots, \mu_d) \in (\mathbb{C}^*)^d \left| \prod_{i=1}^d \mu_i^{\langle m, V_i \rangle} = 1, m \in \mathbb{Z}^3 \right. \right\}, \tag{4.2}$$

which acts on x_i as

$$(x_1, \dots, x_d) \rightarrow (\mu_1 x_1, \dots, \mu_d x_d).$$

K is isomorphic, in general, to $(\mathbb{C}^*)^{d-3}$ times a discrete group. Then, the conical CY is defined by the symplectic quotient:

$$CY = (\mathbb{C}^d \setminus \Delta) / K$$

where Δ is a subset fixed by the action of K . Geometrically, the x_i can be interpreted as homogeneous coordinates on the CY, just like the familiar coordinates for projective spaces. The residual $(\mathbb{C}^*)^3$ complex torus action acting on the CY is dual to the flavor symmetry group in the gauge theory, while the group K is dual to the non-anomalous

baryonic symmetry group. Notice that the flavor and baryonic symmetries nicely combine in the full group of d non-anomalous charges which act naturally on the d homogeneous coordinates x_i as $(\mathbb{C}^*)^d$ acts on \mathbb{C}^d . In the tiling construction, the x_i are used to assign the non-anomalous charges to each field in the quiver [53, 52, 35].

On the other hand, in toric geometry each edge V_i determines a (not necessarily compact) four-cycle D_i in the CY. A generic four-cycle is given by a linear combination of basic divisors $\sum_{i=1}^d a_i D_i$ where a_i are integer coefficients. These divisors are subject to precisely three linear equivalence conditions given by

$$\sum_{i=1}^d \langle e_k, V_i \rangle D_i = 0 \quad k = 1, 2, 3$$

where e_k is a basis for \mathbb{Z}^3 . It is then easy to see that the group of four-cycles modulo linear equivalence is isomorphic to the baryonic group K . It follows that the non-anomalous baryonic symmetry distinguishes deformation equivalence classes of Euclidean D3-branes.

However this is not the end of the story. The decomposition into non-anomalous baryonic charges is not fine enough. A D3-brane state with baryonic charge B can form a sort of bound state which distinguishes it from a set of B D3-brane states with baryonic charge one. This typically happens in theories where there are elementary fields with multiple non-anomalous baryonic charges. By going over examples, it is easy to convince oneself that the classical D3 brane configurations obtained from divisors on the singular CY do not exhaust all possible sectors of the dual gauge theory. However, as already mentioned, we have a plethora of compact vanishing four-cycles that are expected to enter in the description of the set of supersymmetric D3-branes and solve these ambiguities. We have exactly I compact vanishing four-cycles, one for each integer internal point in the toric diagram. These cycles become of finite size in the smooth resolutions of the CY. We will see that with the addition of these divisors we can give a convenient description of all sectors in the dual gauge theory. It would be interesting to understand the necessity for the inclusions of these divisors directly from the point of view of the geometric quantization of classical supersymmetric branes living on the horizon.

We are led to enlarge the set of basic divisors of size $d-3$ to a larger set of size $d-3+I$ by adding a divisor D_i for each internal point of the toric diagram. We now have a larger group of divisors which strictly contains the baryonic symmetry group. The larger set of divisors immediately leads us to the description of the Kähler moduli space of the CY, of dimension $d-3+I$. This moduli space is still a toric variety described by the so-called secondary fan, or GKZ fan, and it is indeed parameterized by the divisors D_i in the larger set.

4.2.1 The GKZ decomposition

It is well known that there are many different smooth resolutions of the CY corresponding to the possible complete triangulations of the toric diagram. Different resolutions are connected by flops. The number of Kähler moduli of the CY is $I+d-3$ where I is the number of internal points; this is the same as the number of geometrical FI terms that appear in the symplectic quotient description of the resolved manifold. This number can

be greater than the number of non-anomalous baryonic symmetries in field theory, which is $d - 3$.

There is an efficient description of the Kähler moduli space in terms of divisors [67]. Take a complete resolution of the variety and consider the set of all effective divisors

$$\left\{ \sum_{i=1}^{d+I} a_i D_i, \text{ such that } a_i \geq 0, \forall i \right\} \quad (4.3)$$

modulo the three linear equivalence conditions given by $\sum_{i=1}^{d+I} \langle e_k, V_i \rangle D_i = 0$ where e_k is the standard basis for \mathbb{Z}^3 and V_i are the vertices of the toric diagram, including the internal integer points. The $a_i \in \mathbb{R}^+$ give a parametrization of the $d + I - 3$ dimensional Kähler moduli provided we impose a further condition: to have all cycles of positive volumes we must consider only *convex* divisors. The convexity conditions can be expressed as follows. Assign a number a_i to the i -th point in the toric diagram. To each triangle σ in the triangulation of the toric diagram we assign a vector $m_\sigma \in \mathbb{Z}^3$ which is the integer solution of the system of three linear equations,⁴

$$\langle m_\sigma, V_i \rangle = -a_i, \quad i \in \sigma \quad (4.4)$$

and impose the inequalities

$$\langle m_\sigma, V_i \rangle \geq -a_i, \quad i \notin \sigma \quad (4.5)$$

The set of inequalities (4.5), as σ runs over all the triangles, determines the convexity condition for the divisor. For a given resolution, the set of convex divisors forms a cone in the \mathbb{R}^{d+I-3} vector space, that parameterizes the Kähler moduli of the resolution. The boundary of this cone corresponds to the vanishing of some cycle. If we can perform a flop, we enter a new region in the moduli space corresponding to a different resolution. Indeed, the cones constructed via the convexity condition for the various possible resolutions of the toric diagram form regions in the \mathbb{R}^{d+I-3} vector space that are adjacent; altogether these reconstruct a collection of adjacent cones (a fan in toric language) in \mathbb{R}^{d+I-3} . The toric variety constructed from this fan in \mathbb{R}^{d+I-3} is the Kähler moduli space of the CY. This is known as the GKZ fan, or secondary fan [68, 69]. We move from a cone in the GKZ fan to another by performing flops (or in case we also consider orbifold resolutions by flops or further subdivisions of the toric diagram). It is sufficient for us to consider smooth varieties and we thus reserve the name *GKZ fan* to the collections of cones corresponding to smooth resolutions.

The GKZ fans for the conifold, \mathbb{F}_0 and dP_1 are given in figures 2, 8 and 12, respectively.

We form a lattice by considering the integer points in the GKZ fan. We claim that the $N = 1$ generating function has an expansion in sectors corresponding to the integer points of the GKZ lattice. Denote by P an integer point in the GKZ lattice, then

$$g_1(\{t_i\}) = \sum_{P \in GKZ} m(P) g_{1,P}(\{t_i\}), \quad (4.6)$$

⁴Actually, these equations can be solved for all simplicial resolutions, corresponding to not necessarily maximal triangulations of the toric diagram. If we allow triangles with area greater than one, we have resolved varieties which still have orbifold singularities. For completely smooth resolutions, the vertices of all triangles σ are primitive vectors in \mathbb{Z}^3 and equation (4.4) has integer solutions.

where $m(P)$ is the multiplicity of the point P . Furthermore, we conjecture that the finite N generating function can be obtained as

$$\sum_N g_N(\{t_i\}) \nu^N = \sum_{P \in GKZ} m(P) \text{PE}_\nu[g_{1,P}(\{t_i\})] \quad (4.7)$$

4.2.2 GKZ and field theory content

At the heart of the previous formulae, there is a remarkable connection between the GKZ decomposition and the quiver gauge theory. To fully appreciate it we suggest to the reader to read this and the following subsections in close parallel with section 5 where explicit examples are given.

The integer points in the GKZ fan correspond to sectors in the quantum field theory Hilbert space made out of determinants. Recall that mesons in the quiver gauge theories correspond to closed paths in the quiver. We want to associate similarly the other independent sectors made out of determinants with equivalence classes of open paths in the dimers. The open paths fall into equivalence classes A specified by the choice of ending points on the dimer. The open path in a given class can be reinterpreted in the gauge theory as strings of elementary fields with all gauge indices contracted except two corresponding to a choice of a specific pair of gauge groups; let us call these *composite fields*. Baryonic operators are written as “ $\det A$ ”, which is a schematic expression for two epsilons contracted with N composite fields freely chosen among the representatives of the class A . Generic sectors are made with arbitrary products $\det A \det B \dots$ etc. Whenever open paths A and B can be composed to give the open path C , there is at least one choice of representatives for A and B such that we can write $\det A \det B = \det C$ and we want to consider the two sectors $\det A \det B$ and $\det C$ equivalent. This can be enforced as follows. Denote with letters $a, b, c \dots$ the equivalence classes of arrows in the quiver connecting different gauge groups. By decomposing open paths in strings of letters, we can associate a sector with a string of letters. We should however take into account the fact that if, for example, the arrows a, b, c make a closed loop, the operator

$$(\det a)(\det b)(\det c) = \det abc$$

is a meson. We take into account this fact by imposing the constraint $abc = 0$. Moreover, composite fields connecting the same pairs of gauge groups as an elementary fields do not determine the existence of new independent determinants; to avoid overcountings, the corresponding string of letters should be set to zero. Analogously, whenever two different strings of letters correspond to open paths with the same endpoints, these strings should be identified. We call \mathcal{I} the set of constraints obtained in this way and construct the ring

$$\mathcal{R}_{\text{GKZ}} = \mathbb{C}[a, b, c \dots] / \mathcal{I}$$

Quite remarkably, the monomials in the ring \mathcal{R}_{GKZ} are in correspondence with the integer points in the GKZ fan. More precisely, we can grade the ring with $d - 3 + I$ charges in such a way that the generating function of \mathcal{R}_{GKZ} , which we call the *auxiliary*

GKZ partition function, and is denoted by Z_{aux} , counts the integer points in the GKZ fan. Moreover, any integer point P comes with a *multiplicity* $m(P)$ which is the one in equation (4.6).

We have explicitly verified the above statement in all the examples we studied and we conjecture that it is a general result for all toric diagrams: the auxiliary partition function counting open paths in the quiver modulo equivalence coincides with the generating function for the GKZ lattice dressed with field theory multiplicities. Just another remarkable connection between apparently different objects: combinatorics on the tiling, geometry of the CY and gauge theory!

As another example of this fascinating correspondence, we will see that it is possible to eliminate multiplicities by refining the GKZ lattice. This is done by enlarging the set of charges. In particular, we have found that, if we refine the GKZ lattice by adding the anomalous baryonic charges, we obtain a “hollow cone” in \mathbb{Z}^{d-3+3I} with no multiplicities.

As previously explained in a generic toric quiver gauge theory there are $2I$ anomalous baryonic symmetries, where I is the number of internal integer points. The variables in the auxiliary GKZ ring \mathcal{R}_{GKZ} corresponds to arrows in the quiver and therefore can be assigned a definite charge under the baryonic symmetries that are just the ungauged $U(1)$ gauge group factors. We can therefore grade the auxiliary ring \mathcal{R}_{GKZ} with a set of $d - 3 + 3I$ weights: the original $d - 3 + I$ discretized Kähler parameters $(\beta_1, \dots, \beta_{d-3+I})$ of the GKZ lattice plus the $2I$ anomalous baryonic weights a_i . The power series expansion of the auxiliary partition function $Z_{\text{aux}}(\beta_i, a_i)$ will draw a lattice in \mathbb{Z}^{d-3+3I} which has the shape of an hollow cone over the GKZ fan: over each point of the GKZ fan there is a hollow polygon $C(P)$ whose shape is related to the pq-web of the toric geometry. Quite remarkably, all points in the lattice come with multiplicity one. Examples for $\mathbb{C}^3/Z_3, \mathbb{F}0$ and dP_1 are presented in figure 5, 9, 13 and 14.

4.2.3 Computing $g_{1,P}$ for one D brane in a sector P using localization

We want to demonstrate now how it is possible to compute the partition functions $g_{1,P}$ using the equivariant index theorem.

Every integer point P in the GKZ fan is associated with a smooth resolution of the CY and a particular divisor $\sum a_i D_i$ on it. Extending our interpretation of the BPS states in terms of supersymmetric D3-branes wrapping holomorphic cycles in the singular CY to its resolution, we have a natural definition for the function $g_{1,P}$: it should count all the sections of the line bundle $\mathcal{O}(\sum a_i D_i)$ corresponding to holomorphic surfaces in the given linear equivalence class. Therefore $g_{1,P}$ is just the character

$$\text{Tr} \left\{ H^0 \left(CY, \mathcal{O} \left(\sum a_i D_i \right) \right) | q \right\}$$

under the action of the element $q \in T^3$ of the torus of flavor symmetries. All elements in $H^0(CY, \mathcal{O}(\sum a_i D_i))$ have the same baryonic charges. It is important to notice that the higher cohomology groups of a convex line bundle vanish [70]: the character then coincides with the holomorphic Lefschetz number and can be computed with the equivariant index theorem.

The way of doing this computation is explained in detail in [9], generalizing the analogous computation for holomorphic functions given in [50], and expresses the result as a sum over the fixed points P_I of the torus T^3 action on the particular smooth resolution of the CY corresponding to the point P in the GKZ lattice. It is known indeed that the torus action has only isolated fixed points on the resolved CY. The character receives contributions only from the fixed points and reads

$$g_{1,P}(\{t_i\}; CY) = t^{n_P} \sum_{P_I} \frac{q^{m_P^{(I)}}}{\prod_{i=1}^3 (1 - q^{m_i^{(I)}})}, \quad (4.8)$$

where the index I denotes the set of isolated fixed points and the four vectors $m_i^{(I)}$, $i = 1, 2, 3$, $m_P^{(I)}$ in \mathbb{Z}^3 are the weights of the linearized action of T^3 on the resolved CY and the fiber of the line bundle $\mathcal{O}(\sum a_i D_i)$, respectively.

The fixed points of the torus action are in correspondence with the triangles in the subdivision of the toric diagram (or, equivalently, with the vertices of the pq-web). The vectors $m_i^{(I)}$, $i = 1, 2, 3$ in the denominator of equation (4.8) are computed as the three primitive inward normal vectors of the cone σ_I in \mathbb{Z}^3 made with the three vertices V_i of the I -th triangle. The vector $m_0^{(I)}$ in the numerator is instead computed as in equation (4.4)

$$\langle m_0^{(I)}, V_i \rangle = -a_i, \quad i \in \sigma_I \quad (4.9)$$

Finally, the prefactor t^{n_P} in equation (4.8) is just the charge of the divisor $\sum a_i D_i$. The full dependence on baryonic charges is encoded in this prefactor.

In explicit computations, some care should be paid to the choice of charges. There is a natural geometric basis for the non-anomalous charges of the gauge theory. In fact, the homogeneous coordinates x_i that are used to define the CY as a symplectic quotient (see section 4.2.1) are extremely useful to assign a full set of d (flavor+baryonic) charges to the elementary fields in the quiver; this is done using zig-zag paths and standard dimers techniques [35, 55]. All the elementary fields have charge which is given by a product of the x_i . We can also assign charge x_i to the divisor D_i of the singular cone. In all the examples we have considered there is a natural way to assign charges to the enlarged set of divisors entering the GKZ decomposition. This allows to compute the prefactor t^{n_P} . The x_i decompose into three flavor charges and $d-3$ baryonic charges. The splitting of the charges x_i into flavor and baryonic charges is not unique in general; flavor charges can be always redefined by adding a linear combination of the baryonic charges. However, a toric diagram comes with a specific basis for the flavor T^3 action which is determined by the equation

$$q_k = \prod_{i=1}^d x_i^{\langle e_k, V_i \rangle} \quad k = 1, 2, 3, \quad (4.10)$$

where e_k are the basis vectors of \mathbb{Z}^3 and V_i the vertices of the toric diagram. Notice that all dependence on baryonic charges drops from the right hand side by equation (4.2). This is the T^3 basis that should be used in the localization formula (4.8).

4.2.4 Checks with all charges: GKZ approach vs. field theory

Having understood how to compute and resolve the multiplicities in the GKZ cones and to compute the partition functions we can also refine our decomposition of the $N = 1$ generating function by adding the anomalous baryonic charges.

Using the equivariant index theorem we compute all the generating functions $g_{1,P}$ for all points P in the GKZ lattice. We can use $d - 3 + I$ coordinates for the GKZ cone $\beta = (\beta_1, \dots, \beta_{d-3+I})$. Denote also with $B(\beta)$ the non-anomalous baryonic charge corresponding to the point P of the GKZ lattice. As discussed in section 4.2.3, the generating functions depend on the baryonic charges only by a multiplicative factor: $g_{1,P} = b^{B(\beta)} g_{1,\beta}(q)$ and all the other dependence is on the flavor charges q_i . Thanks to the auxiliary generating functions we were able to find expressions for the multiplicities $m(\beta)$ of the fields over each point of the GKZ lattice. These functions sum up to the complete generating functions with $N = 1$ and with all the non-anomalous charges:

$$g_1(q, b) = \sum_{P \in GKZ} m(P) g_{1,\beta}(q) b^{B(\beta)} \tag{4.11}$$

where b are the chemical potentials for the non-anomalous baryonic charges. To resolve the multiplicities $m(\beta)$ we construct the hollow cone by adding the anomalous baryonic charges. Over each point in the GKZ lattice there is a hollow polygon $C(\beta)$ which can be parametrized in terms of the set of $2I$ anomalous charges a_j with $j = 1, \dots, 2I$, such that:

$$\sum_{K_j \in C(\beta)} a_1^{K_1} \dots a_{2I}^{K_{2I}} \Big|_{(a_1=1, \dots, a_{2I}=1)} = m(P) \tag{4.12}$$

Using these resolutions we obtain the resolved generating functions for $N = 1$ with all the charges, anomalous and non-anomalous:

$$g_1(q, b, a) = \sum_{\beta \in GKZ} \sum_{K_j \in C(\beta)} a_1^{K_1} \dots a_{2I}^{K_{2I}} b^{B(\beta)} g_{1,\beta}(q) \tag{4.13}$$

The non flavor charges do not appear in the basic generating functions $g_{1,\beta}(q)$, but they are multiplicative factors over which one has to sum up, in the same way one does for the usual non-anomalous baryonic charges.

We would like to stress that equation (4.13) points to a remarkable connection between the geometry of the CY, which is used to compute the right hand side, and the field theory, that can be used to determine the left hand side (the $N = 1$ generating function). In other words, we have two different ways of computing the $N = 1$ generating $g_1(q, b, a)$ function which nicely match:

- In the first case we use the GKZ geometric picture explained in the previous sections. We first compute the generating functions $g_{1,\beta}(q)$ for each point in the GKZ lattice which depend on the flavor charges. We next sum over all the points of the hollow cone by dressing $g_{1,\beta}(q)$ with the appropriate weight under the baryonic symmetries.

- In the second case we use the field theory picture. We can take the fields of the gauge theory as basic variables and we assign to them all the possible charges anomalous and non-anomalous. This means that we construct the ring generated by the fundamental fields and we grade it with all the charges. Then we construct the quotient ring by modding out the ring of elementary fields by the ideal generated by F -term equations. Using Macaulay2 we compute the Hilbert series of the quotient ring obtaining the completed resolved generating function with all the charges of the field theory: $g_1(q, b, a)$.

As we will explicitly demonstrate on the examples in section 5, the two computations completely agree.

As a final remark, we notice that equation (4.13) is the most general decomposition of the $N = 1$ generating function that we can write. We can close our circular discussion and go back to the initial point. Equation (4.13) has been obtained by enlarging the GKZ lattice in order to eliminate multiplicities. The hollow cone is a lattice in $d - 3 + 3I$ dimensions. The corresponding $d - 3 + 3I$ charges contain, as a subset, all the anomalous and non-anomalous baryonic charges that are in number $d - 3 + 2I$. Notice that the terms in series in equation (4.13) depend on the extra I GKZ parameters only through the factor $b^{B(\beta)}$. By projecting the hollow cone on the $d - 3 + 2I$ dimensional space of baryonic charges we obtain the explicit expansion of the $N = 1$ generating function $g_1(q, b, a)$ in a complete set of baryonic charges which was discussed in section 4.1. One can compare this expansion with the one obtained by performing repeated contour integrations. As one can check explicitly, the points in the baryonic charge lattice have still multiplicity one.

5. Examples

In this section we explicitly compute the $N = 1$ generating function for a certain number of toric CY manifolds and decompose it. We start by revisiting the example of the conifold.

5.1 The conifold revisited

The conifold has only one baryonic charge, not anomalous, which can be used to parametrize the Kähler moduli space. The two expansions, one in baryonic charges, the other according to the GKZ lattice, coincide.

Baryonic charge expansion. We first expand the $N = 1$ generating function, equation (3.1) for the conifold according to the baryonic charge

$$g_1(t, b, x, y; \mathcal{C}) = \sum_{B=-\infty}^{\infty} b^B g_{1,B}(t, x, y; \mathcal{C}), \tag{5.1}$$

$g_{1,B}(t, x, y; \mathcal{C})$ can be computed using the inversion formula

$$g_{1,B}(t, x, y; \mathcal{C}) = \frac{1}{2\pi i} \oint \frac{db}{b^{B+1}} g_1(t, b, x, y; \mathcal{C}), \tag{5.2}$$

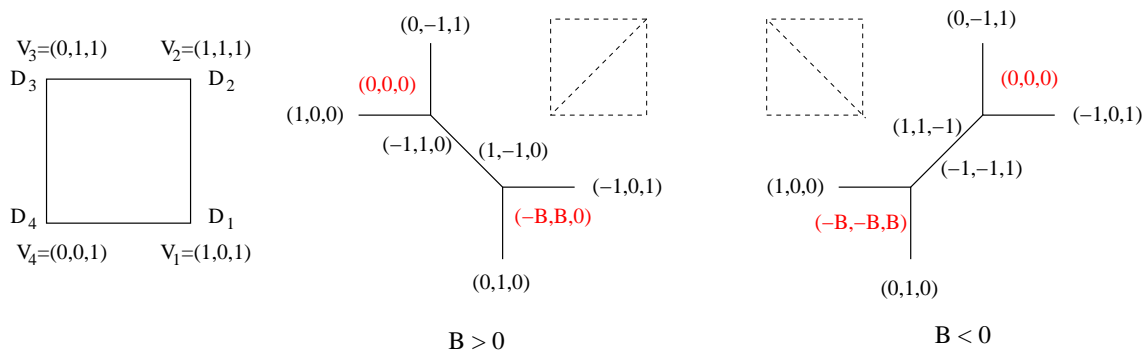


Figure 3: Localization data for the $N = 1$ baryonic generating functions. The vertices V_i are in correspondence with homogeneous coordinates x_i and with a basis of divisors D_i . Two different resolutions, related by a flop, should be used for positive and negative B , respectively. Each resolution has two fixed points, corresponding to the vertices of the pq-webs; the weights $m_i^{(I)}$, $i = 1, 2, 3$ and $m_B^{(I)}$ at the fixed points are indicated in black and red, respectively.

with a careful evaluation of the contour integral for positive and negative values of the baryonic charge B . For $B \geq 0$ the contribution of the contour integral comes from the positive powers of the poles for b ($b = x/t, 1/(xt)$) while for $B \leq 0$ the contribution of the contour integral comes from the negative powers of the poles for b ($b = ty, t/y$)

$$\begin{aligned}
 g_{1,B \geq 0}(t, x, y; \mathcal{C}) &= \frac{t^B x^B}{\left(1 - \frac{1}{x^2}\right) (1 - t^2 xy) \left(1 - \frac{t^2 x}{y}\right)} + \frac{t^B x^{-B}}{(1 - x^2) \left(1 - \frac{t^2 y}{x}\right) \left(1 - \frac{t^2}{xy}\right)}, \\
 g_{1,B \leq 0}(t, x, y; \mathcal{C}) &= \frac{t^{-B} y^{-B}}{\left(1 - \frac{1}{y^2}\right) (1 - t^2 xy) \left(1 - \frac{t^2 y}{x}\right)} + \frac{t^{-B} y^B}{(1 - y^2) \left(1 - \frac{t^2 x}{y}\right) \left(1 - \frac{t^2}{yx}\right)}.
 \end{aligned}
 \tag{5.3}$$

By setting $x = y = 1$ and $t_1 = bt, t_2 = t/b$ we recover expansion (3.2).

5.1.1 Conifold-GKZ decomposition

We can similarly perform a GKZ decomposition of the $N = 1$ generating function. In figure 3 the toric diagram and the two resolutions of the conifold are reported. There are four divisors D_i subject to three relations that leave an independent divisor D , $D_1 = D_3 = -D_2 = -D_4 \equiv D$. Consider the cone of effective divisors $\sum a_i D_i$, $a_i \geq 0$ modulo linear equivalence in \mathbb{R}

$$\sum_{i=1}^4 a_i D_i \equiv (a_1 + a_3 - a_2 - a_4) D \equiv B D$$

where we defined $B = a_1 + a_3 - a_2 - a_4$. For each resolution, we solve equation (4.4) for the two triangles in the resolution, or, equivalently, the two vertices of the pq-web; the resulting vectors $m_i^{(I)}$ and $m_B^{(I)}$ are reported in black and red respectively in figure 3.

The convexity condition, equation (4.5), then tells us that the resolution on the left corresponds to $B > 0$ and the resolution on the right to $B < 0$. Altogether we obtain two

half lines (cones) in \mathbb{R} that form the GKZ fan as in figure 2. The point of intersection $B = 0$ of the two cones corresponds to the singular conifold and the two cones in the fan are related by a flop transition.

We now compute the generating functions $g_{1,B}$ using localization. As mentioned in the previous section, we must pay attention to the normalization of charges. The homogeneous coordinates for the conifold are extremely simple:

$$(\mathbf{A}_1, \mathbf{B}_1, \mathbf{A}_2, \mathbf{B}_2) \longrightarrow (x_1, x_2, x_3, x_4)$$

which can be easily translated in the notations of section 3. The natural flavor T^3 basis is then given by equation (4.10)

$$\begin{aligned} q_1 &= x_1 x_2 = t^2 x y, \\ q_2 &= x_2 x_3 = \frac{t^2 y}{x}, \\ q_3 &= x_1 x_2 x_3 x_4 = t^4. \end{aligned} \tag{5.4}$$

We are ready to apply the localization formula. Each point in the GKZ fan is associated with a resolution and a divisor: for $B > 0$ we use the resolution on the left in figure 3 and BD_1 , while for $B < 0$ the resolution on the right and the divisor $|B|D_4$. The weights are reported in figure 3. Equation (4.8) and equation (5.4) give

$$\begin{aligned} g_{1,B \geq 0}(t, x, y; \mathcal{C}) &= \frac{t^B x^B}{\left(1 - \frac{1}{x^2}\right) (1 - t^2 x y) \left(1 - \frac{t^2 x}{y}\right)} + \frac{t^B x^{-B}}{(1 - x^2) \left(1 - \frac{t^2 y}{x}\right) \left(1 - \frac{t^2}{xy}\right)}, \\ g_{1,B \leq 0}(t, x, y; \mathcal{C}) &= \frac{t^{-B} y^{-B}}{\left(1 - \frac{1}{y^2}\right) (1 - t^2 x y) \left(1 - \frac{t^2 y}{x}\right)} + \frac{t^{-B} y^B}{(1 - y^2) \left(1 - \frac{t^2 x}{y}\right) \left(1 - \frac{t^2}{yx}\right)}. \end{aligned}$$

which coincides with the result previously obtained in equation (5.3).

5.1.2 Conifold-multiplicities in the GKZ decomposition

The multiplicities in the GKZ decomposition of the $N = 1$ generating function for the conifold are trivial since they are all equal to 1. Nevertheless it is instructive to follow the procedure which is outlined in section 4.2.2 in order to compute the multiplicities using the auxiliary GKZ partition function $Z_{\text{aux}}(t)$ which counts independent sectors in the ring of invariants. As explained in section 4.2.2, we assign a letter a, b to the two types of arrows in the quiver $\mathbf{A}_i, \mathbf{B}_i$. There is only one relation $ab = 0$ corresponding to the closed loop in the quiver. The polynomial ring for the GKZ decomposition of the conifold is therefore

$$\mathcal{R}_{\text{GKZ}}(\mathcal{C}) = \mathbb{C}[a, b]/(ab), \tag{5.5}$$

We thus compute the generating function for the polynomial ring (which can be easily computed by observing it is a complete intersection), with chemical potential t_1 to a and t_2 to b we find

$$Z_{\text{aux}}(t_1, t_2; \mathcal{C}) = \frac{1 - t_1 t_2}{(1 - t_1)(1 - t_2)} = 1 + \sum_{B=1}^{\infty} t_1^B + \sum_{B=1}^{\infty} t_2^B \tag{5.6}$$

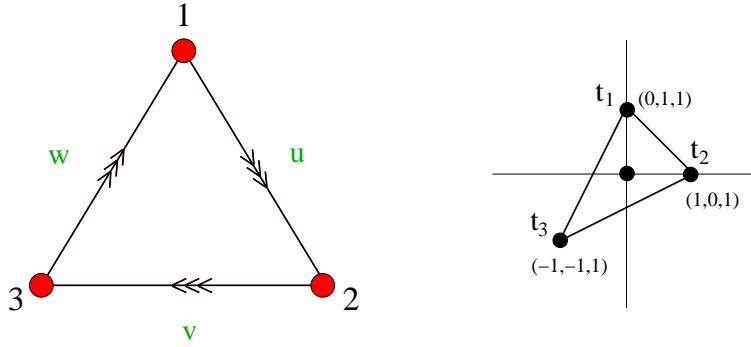


Figure 4: Quiver and toric diagram for $\mathbb{C}^3/\mathbb{Z}_3$.

By expanding this auxiliary partition function we find multiplicity 1 for the integer points $B > 0$, multiplicity 1 for the integer points $B < 0$, and multiplicity 1 for the point $B = 0$, reproducing the lattice depicted in figure 2. We finally have

$$g_1(t_1, t_2) = g_{1,0}(t_1, t_2) + \sum_{B=1}^{\infty} g_{1,B}(t_1, t_2) + \sum_{B=-\infty}^{-1} g_{1,B}(t_1, t_2), \quad (5.7)$$

which appears to be a trivial observation as a Laurent series in the baryonic chemical potential but in fact turns out to be nontrivial for more involved singularities.

5.2 Generating functions for $\mathbb{C}^3/\mathbb{Z}_3$

The quiver of the gauge theory for the $\mathbb{C}^3/\mathbb{Z}_3$ singularity is shown in figure 4. The gauge theory has three sets of bifundamental fields $\mathbf{U}_i, \mathbf{V}_j, \mathbf{W}_k$ with $i, j, k = 1, 2, 3$ and a superpotential $\epsilon_{ijk} \mathbf{U}_i \mathbf{V}_j \mathbf{W}_k$.

Symmetries and geometry. The global flavor symmetry is $SU(3) \times U(1)_R$. All the fields have R-charge $2/3$ and each set of three fields transform in the fundamental representation of $SU(3)$. One can also define two anomalous baryonic $U(1)$ charges which can be chosen to be

$$\begin{aligned} A_1 : (\mathbf{U}, \mathbf{V}, \mathbf{W}) &\mapsto (a_1 \mathbf{U}, a_1^{-1} \mathbf{V}, \mathbf{W}) \\ A_2 : (\mathbf{U}, \mathbf{V}, \mathbf{W}) &\mapsto (a_2 \mathbf{U}, a_2 \mathbf{V}, a_2^{-2} \mathbf{W}) \end{aligned}$$

There are some non-anomalous discrete symmetries [71] acting on the fields as follows,

$$\begin{aligned} A : (\mathbf{U}, \mathbf{V}, \mathbf{W}) &\mapsto (\mathbf{W}, \mathbf{U}, \mathbf{V}) \\ B : (\mathbf{U}, \mathbf{V}, \mathbf{W}) &\mapsto (b\mathbf{U}, b^{-1}\mathbf{V}, \mathbf{W}) \quad \text{where } b^3 = 1 \\ C : (\mathbf{U}, \mathbf{V}, \mathbf{W}) &\mapsto (c\mathbf{U}, c\mathbf{V}, c^{-2}\mathbf{W}) = (c\mathbf{U}, c\mathbf{V}, c\mathbf{W}) \quad \text{where } c^3 = 1 \end{aligned}$$

We see that B is a subgroup of A_1 and C is subgroup of A_2 . C is related to the torsion homology group for three cycles and, in a sense, is a discrete baryonic charge.

field	SU(3)	R	A ₁	A ₂	chemical potentials	homogeneous coordinates
(U ₁ , U ₂ , U ₃)	3	$\frac{2}{3}$	1	1	(ta ₁ a ₂ y, ta ₁ a ₂ x, ta ₁ a ₂ /xy)	(ct ₁ , ct ₂ , ct ₃)
(V ₁ , V ₂ , V ₃)	3		-1	1	(ta ₂ y/a ₁ , ta ₂ x/a ₁ , ta ₂ /a ₁ xy)	(ct ₁ , ct ₂ , ct ₃)
(W ₁ , W ₂ , W ₃)	3		0	-2	(ty/a ₂ ² , tx/a ₂ ² , t/a ₂ ² xy)	(ct ₁ , ct ₂ , ct ₃)

Table 2: Global charges for the basic fields of the quiver gauge theory living on the D-brane probing the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. The x and y chemical potentials count SU(3) weights, while A_1 and A_2 count anomalous baryonic charges.

We now have a look at the geometry of the CY using the symplectic quotient construction outlined in section 4.2. Since $d = 3$ we introduce three homogeneous coordinates x_i . The group K of baryonic charges is defined by equation (4.2)

$$\prod_{j=1}^3 \mu_j^{\langle e_i, V_j \rangle} = 1$$

which implies $\mu_j = c$ with $c^3 = 1$. The symplectic quotient description of the CY just reduces to the orbifold description $\mathbb{C}[x_1, x_2, x_3]/\mathbb{Z}_3$, as expected.

As already discussed, the homogeneous coordinates can be used to give a full set of weights for non-anomalous symmetries to the elementary fields. We can write $x_i = ct_i$ in terms of the discrete baryonic charge c and the flavors t_i . The assignment of charges to the fields is done using standard dimer techniques and is reported in table 2. One can notice that t_i are the charges of the original $\mathcal{N} = 4$ SYM. To keep track of the coordinates on the two dimensional projection in figure 4 we introduce three chemical potentials t, x, y which count the R -charge, and the (x, y) integral positions, respectively and read from the figure $t_1 = ty, t_2 = tx, t_3 = \frac{t}{xy}$.

The full set of continuous charges, anomalous or not, is summarized in the table 2, as well as the chemical potentials and the assignment of homogeneous coordinates to the fields.

5.2.1 The $N = 1$ generating function

The $N = 1$ generating function is generated by the elementary fields U_i, V_j, W_k modulo nine F-term relations which can be expressed through the ideal

$$\begin{aligned} \mathcal{I} = & (\mathbf{V}_2 \mathbf{W}_3 - \mathbf{V}_3 \mathbf{W}_2, \mathbf{V}_1 \mathbf{W}_3 - \mathbf{V}_3 \mathbf{W}_1, \mathbf{V}_1 \mathbf{W}_2 - \mathbf{V}_2 \mathbf{W}_1, \\ & \mathbf{U}_2 \mathbf{W}_3 - \mathbf{U}_3 \mathbf{W}_2, \mathbf{U}_1 \mathbf{W}_3 - \mathbf{U}_3 \mathbf{W}_1, \mathbf{U}_1 \mathbf{W}_2 - \mathbf{U}_2 \mathbf{W}_1, \\ & \mathbf{V}_2 \mathbf{U}_3 - \mathbf{V}_3 \mathbf{U}_2, \mathbf{V}_1 \mathbf{U}_3 - \mathbf{V}_3 \mathbf{U}_1, \mathbf{V}_1 \mathbf{U}_2 - \mathbf{V}_2 \mathbf{U}_1). \end{aligned}$$

Each field carries R-charge $\frac{2}{3}$ and therefore we can give to all the same weight for the chemical potential, t . Computing the Hilbert series of the polynomial ring

$$\mathcal{R}_{N=1}(\mathbb{C}^3/\mathbb{Z}_3) = \mathbb{C}\{\{U_i\}, \{V_j\}, \{W_k\}\}/\mathcal{I} \tag{5.8}$$

with Macaulay2 we obtain

$$g_1(t; \mathbb{C}^3/\mathbb{Z}_3) = \frac{1 + 4t + t^2}{(1 - t)^5}. \tag{5.9}$$

The dimension of an irreducible algebraic variety V can be computed from its Hilbert series $g(t)$ by looking at the order of the pole for $t \rightarrow 1$

$$g_1(t) \sim \frac{A}{(1-t)^{\dim V}}, \tag{5.10}$$

with the residue A a measure for the volume of this variety. The $N = 1$ moduli space of vacua for $\mathbb{C}^3/\mathbb{Z}_3$ has dimension five, as can be seen from

$$g_1(t) \sim \frac{6}{(1-t)^5}$$

This can be understood as three mesonic directions describing the CY plus two independent baryonic parameters that correspond to the gauge theory FI terms. As usual, the mesonic moduli space for $N = 1$ is isomorphic to the CY geometry. The additional baryonic parameters come from the fact that the gauge group is $SU(N)^G$ and not $U(N)^G/U(1)$. Since we do not have to impose the $U(1)$ D-term conditions, this leaves $G - 1$ additional free parameters that can be identified with the FI terms in the gauge theory. Since in general the mesonic flat directions are given by the symmetrized product of N CY's for N D-branes, giving $3N$ parameters, the dimension of the moduli space for generic N and G is expected to be $3N + G - 1$.

5.2.2 The GKZ decomposition

In the singular CY there is only one independent divisor $D_i \equiv D$ with $3D = 0$. This reflects the \mathbb{Z}_3 discrete charge. However, on the smooth resolution of the orbifold there is a new divisor D_4 corresponding to the internal point. $D_1 = D_2 = D_3 = D$ is still true but now $3D$ is non-zero, but equal instead to $-D_4$. The cone of effective divisors in \mathbb{R} is given by

$$\sum_{i=1}^4 a_i D_i = (a_1 + a_2 + a_3 - 3a_4) D \equiv \beta D, \quad a_i \geq 0$$

and the convexity condition, equation (4.5), requires $\beta \geq 0$. The GKZ fan is thus a half-line in \mathbb{R} . The integer parameter β turns out to be the discrete Kähler modulus of the resolution of $\mathbb{C}^3/\mathbb{Z}_3$, measuring the discrete area of the two cycle.

The right basis for localization is given by

$$q_i = \prod_{j=1}^3 x_j^{\langle e_i, V_j \rangle}$$

and we compute (notice that the discrete baryonic charge c correctly drops out from this formula) $q_1 = \frac{t_2}{t_3}, q_2 = \frac{t_1}{t_3}, q_3 = t_1 t_2 t_3$. We thus obtain⁵

$$g_{1,\beta}(t_1, t_2, t_3) = \frac{t_1^\beta}{(1-t_1^3) \left(1 - \frac{t_2}{t_1}\right) (1-t_3/t_1)} + \frac{t_2^\beta}{(1-t_1/t_2)(1-t_2^3)(1-t_3/t_2)} + \frac{t_3^\beta}{(1-t_1/t_3)(1-t_2/t_3)(1-t_3^3)}. \tag{5.11}$$

⁵These partition functions reduce for $\beta = 0, 1, 2$ to the three independent partition functions for nontrivial divisors on the singular cone.

$g_{1,0}$ is identified with the Molien invariant for the discrete group \mathbb{Z}_3 and indeed computes the mesonic generating function as explained in detail in [6]. In the limit $t_i = t$ we find

$$g_{1,\beta}(t, t, t) = t^\beta \left(\frac{1 + 7t^3 + t^6}{(1 - t^3)^3} + \frac{3\beta(1 + t^3)}{2(1 - t^3)^2} + \frac{\beta^2}{2(1 - t^3)} \right). \quad (5.12)$$

5.2.3 Multiplicities

The multiplicities in the GKZ decomposition of the $N = 1$ generating function can be computed using $Z_{\text{aux}}(t)$, the auxiliary GKZ partition function counting independent sectors in the ring of invariants. As explained in section 4.2.2, we assign a letter u, v, w to the three types of arrows in the quiver $\mathbf{U}_i, \mathbf{V}_j, \mathbf{W}_k$. There is only one relation $uvw = 0$ corresponding to the closed loop in the quiver. We thus get the polynomial ring

$$\mathcal{R}_{\text{GKZ}}(\mathbb{C}^3/\mathbb{Z}_3) = \mathbb{C}[u, v, w]/(uvw), \quad (5.13)$$

and compute the generating function (which can be easily computed by assuming it is a complete intersection), with charge t to all letters obtaining

$$Z_{\text{aux}}(t; \mathbb{C}^3/\mathbb{Z}_3) = \frac{1 - t^3}{(1 - t)^3} = 1 + \sum_{\beta=1}^{\infty} 3\beta t^\beta. \quad (5.14)$$

By expanding this auxiliary partition function we find multiplicity 3β for the point $\beta > 0$ and multiplicity 1 for the point $\beta = 0$. This is easily understood: the independent sectors contain determinants of the form $(\det \mathbf{U})^n (\det \mathbf{V})^m$ with $n + m = \beta$ or similar with $\mathbf{U}, \mathbf{V}, \mathbf{W}$ permuted; there are 3β such sectors. This point will be further elaborated below.

We finally have

$$g_1(t_1, t_2, t_3) = g_{1,0}(t_1, t_2, t_3) + \sum_{\beta=1}^{\infty} 3\beta g_{1,\beta}(t_1, t_2, t_3). \quad (5.15)$$

This can be summed easily using equations (5.11) and (5.14) and gives

$$g_1(t_1, t_2, t_3; \mathbb{C}^3/\mathbb{Z}_3) = \frac{1}{(1 - t_1)^3 \left(1 - \frac{t_2}{t_1}\right) (1 - t_3/t_1)} + \frac{1}{(1 - t_1/t_2)(1 - t_2)^3(1 - t_3/t_2)} + \frac{1}{(1 - t_1/t_3)(1 - t_2/t_3)(1 - t_3)^3}. \quad (5.16)$$

For the special case $t_1 = t_2 = t_3 = t$ we can take the limit or resum, using equation (5.12),

$$g_1(t, t, t; \mathbb{C}^3/\mathbb{Z}_3) = \frac{1 + 4t + t^2}{(1 - t)^5}$$

which is exactly equation (5.9).

5.2.4 Refining the GKZ decomposition

Using equation (5.14) we summarize the multiplicities

$$m(\beta) = \begin{cases} 1 & \text{for } \beta = 0 \\ 3\beta & \text{for } \beta > 0. \end{cases} \quad (5.17)$$

For a dibaryon, the AdS/CFT dual object is a D3-brane that wraps a $\Sigma_3 = S^3/\mathbb{Z}_3$ cycle⁶ in S^5/\mathbb{Z}_3 . The homology tells us that the wrapping number is characterized by an integer, modulo 3. By resolving the singular Calabi-Yau, this gets promoted to a (non-negative) integer which is just the coordinate β in the GKZ cone. The GKZ fan does not take into account the possible topologically nontrivial flat connections on the wrapped D3-brane. To avoid multiplicities, we include the U(1) extensions of all the discrete charges. The R-charge is already a coordinate in the GKZ fan and its corresponding GKZ auxiliary generating function is given in equation (5.14). The remaining charges are the anomalous charges A_1 and A_2 , as given in table 2, which we now add to the lattice as extra coordinates. The points in the resulting lattice form a “hollow cone” and have no multiplicities.

The dressed auxiliary GKZ partition function which now also contains the anomalous charges can be computed using the assumption that the polynomial ring, equation (5.13), is a complete intersection,

$$\begin{aligned} Z_{\text{aux}}(t, a_1, a_2; \mathbb{C}^3/\mathbb{Z}_3) &= \frac{1 - t^3}{(1 - ta_1a_2)(1 - ta_2/a_1)(1 - t/a_2^2)} = \\ &= 1 + (a_2^{-2} + a_1^{-1}a_2 + a_1a_2)t + \\ &\quad + (a_2^{-4} + a_1^{-1}a_2^{-1} + a_1a_2^{-1} + a_2^2 + a_1^{-2}a_2^2 + a_1^2a_2^2)t^2 + \\ &\quad + (a_1^{-2} + a_1^2 + a_2^{-6} + a_1^{-1}a_2^{-3} + a_1a_2^{-3} + a_1^{-3}a_2^3 + a_1^{-1}a_2^3 + a_1a_2^3 + a_1^3a_2^3)t^3 + \dots \end{aligned} \quad (5.18)$$

By drawing the lattice points in the (A_1, A_2) lattice one can see that there is a “hollow triangle” $C(\beta)$ above each point β in the 1d GKZ cone (figure 5), with edge length measured by the R-charge. This gives the 1, 3, 6, 9, ... multiplicities. The same triangle appears in the pq-web (figure 6) of the geometry. This is a general feature as we will see in other examples. The polygon in the fiber parameterized by the anomalous charges nicely matches the shape of the blown-up cycle in the pq-web.

We can now refine the decomposition (5.15) by introducing the anomalous charges. We first write the expansion (5.18) in the form

$$Z_{\text{aux}}(t, a_1, a_2; \mathbb{C}^3/\mathbb{Z}_3) = \sum_{\beta=0}^{\infty} \left(\sum_{K \in C(\beta)} a_1^{K_1} a_2^{K_2} \right) t^\beta$$

where the two-dimensional index $K = (K_1, K_2)$ runs over the points of the hollow triangle $C(\beta)$.

⁶Generically, Σ_3 is a Lens space.

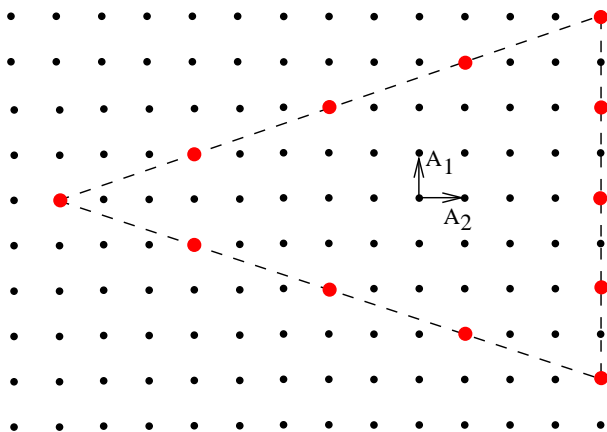


Figure 5: The hollow triangle $C(4)$ above $R = 4$, i. e. the terms containing t^4 . It gives the multiplicity $4 \times 3 = 12$.

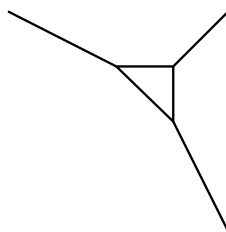


Figure 6: The pq-web of $\mathbb{C}^3/\mathbb{Z}_3$. The blown-up four-cycle is associated to the triangle in the middle.

We can then refine the decomposition (5.15) by replacing the multiplicity $m(\beta)$ with $\sum_{K \in C(\beta)} a_1^{K_1} a_2^{K_2}$,

$$g_1(t_1, t_2, t_3, a_1, a_2) = \sum_{\beta=0}^{\infty} \left(\sum_{K \in C(\beta)} a_1^{K_1} a_2^{K_2} \right) g_{1,\beta}(t_1, t_2, t_3). \quad (5.19)$$

By explicit computation we can resum the previous series and compare with the expected field theory result, finding perfect agreement. The left hand side of formula (5.19) is indeed the $N = 1$ generating function depending on all the five chemical potentials, which can be computed as the Hilbert series for the polynomial ring, equation (5.8) using the grading in table 2,

$$g_1(t_1, t_2, t_3, a_1, a_2; \mathbb{C}^3/\mathbb{Z}_3) = \frac{P(t_1, t_2, t_3, a_1, a_2)}{\left(1 - \frac{t_1}{a_2}\right) \left(1 - \frac{a_2 t_1}{a_1}\right) (1 - a_1 a_2 t_1) \left(1 - \frac{t_2}{a_2}\right) \left(1 - \frac{a_2 t_2}{a_1}\right) (1 - a_1 a_2 t_2) \left(1 - \frac{t_3}{a_2}\right) \left(1 - \frac{a_2 t_3}{a_1}\right) (1 - a_1 a_2 t_3)} \quad (5.20)$$

where $P(t_1, t_2, t_3, a_1, a_2)$ is a polynomial in the gauge theory chemical potentials

$$\begin{aligned}
 P(t_1, t_2, t_3, a_1, a_2) = & 1 - \frac{t_1 t_2}{a_1 a_2} - \frac{a_1 t_1 t_2}{a_2} - a_2^2 t_1 t_2 + t_1^2 t_2 + t_1 t_2^2 - \frac{t_1 t_3}{a_1 a_2} - \frac{a_1 t_1 t_3}{a_2} - a_2^2 t_1 t_3 \\
 & + t_1^2 t_3 - \frac{t_2 t_3}{a_1 a_2} - \frac{a_1 t_2 t_3}{a_2} - a_2^2 t_2 t_3 + 4 t_1 t_2 t_3 + \frac{t_1 t_2 t_3}{a_1^2} + a_1^2 t_1 t_2 t_3 \\
 & + \frac{t_1 t_2 t_3}{a_1 a_2^3} + \frac{a_1 t_1 t_2 t_3}{a_2^3} + \frac{a_2^3 t_1 t_2 t_3}{a_1} + a_1 a_2^3 t_1 t_2 t_3 - \frac{t_1^2 t_2 t_3}{a_2^2} - \frac{a_2 t_1^2 t_2 t_3}{a_1} \\
 & - a_1 a_2 t_1^2 t_2 t_3 + t_2^2 t_3 - \frac{t_1 t_2^2 t_3}{a_2^2} - \frac{a_2 t_1 t_2^2 t_3}{a_1} - a_1 a_2 t_1 t_2^2 t_3 + t_1 t_3^2 + t_2 t_3^2 \\
 & - \frac{t_1 t_2 t_3^2}{a_2^2} - \frac{a_2 t_1 t_2 t_3^2}{a_1} - a_1 a_2 t_1 t_2 t_3^2 + t_1^2 t_2^2 t_3^2
 \end{aligned} \tag{5.21}$$

We would like to stress that decomposition (5.19) is highly nontrivial. The right hand side has been computed from the geometrical localization formulae and the refined GKZ auxiliary generating function. It is then remarkable that the sum on the right hand side coincides with the field theory $N = 1$ generating function.

Using equations (5.11) and (5.18) we get the following simpler expression

$$\begin{aligned}
 g_1(t_1, t_2, t_3, a_1, a_2; \mathbb{C}^3/\mathbb{Z}_3) = & \frac{1}{(1 - t_1 a_1 a_2) \left(1 - \frac{t_1 a_2}{a_1}\right) \left(1 - \frac{t_1}{a_2^2}\right) \left(1 - \frac{t_2}{t_1}\right) (1 - t_3/t_1)} \\
 & + \frac{1}{(1 - t_1/t_2)(1 - t_2 a_1 a_2)(1 - t_2 a_2/a_1)(1 - t_2/a_2^2)(1 - t_3/t_2)} \\
 & + \frac{1}{(1 - t_1/t_3)(1 - t_2/t_3)(1 - t_3 a_1 a_2)(1 - t_3 a_2/a_1)(1 - t_3/a_2^2)}.
 \end{aligned} \tag{5.22}$$

The previous formula suggests the existence of a localization formula for the holomorphic functions on the $N = 1$ moduli space, which is a five-dimensional variety with an action of five $U(1)$ symmetries, three flavor plus two baryonic.

By projecting the refined GKZ expansion on the plane (A_1, A_2) we would get the expansion of the $N = 1$ generating functions into sectors with definite baryonic charge. The same result can be obtained by expanding g_1 in a Laurent series by means of the residue theorem. It is easy to check that the multiplicity of each sector is one.

5.2.5 Generating functions for $N > 1$

The generating function g_N is now obtained from the general formula (2.2) starting from *any* decomposition of the $N = 1$ generating function, the GKZ decomposition (5.15), the refined GKZ decomposition (5.19) or the anomalous baryonic charge decomposition. Since we are interested in writing generating functions depending on the non-anomalous charges, at the end of the computation a_i should be set to one.

The more economical way of obtaining g_N is to start from decomposition (5.15). The generating function for N D-branes is now given by the plethystic exponentiation

$$\sum_{N=0}^{\infty} g_N(t_1, t_2, t_3) \nu^N = \text{PE}_{\nu}[g_{1,0}(t_1, t_2, t_3)] + \sum_{\beta=1}^{\infty} 3\beta \text{PE}_{\nu}[g_{1,\beta}(t_1, t_2, t_3)] \tag{5.23}$$

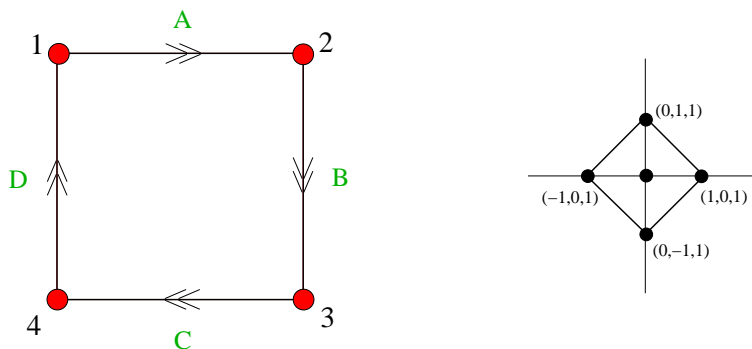


Figure 7: Quiver and toric diagram for \mathbb{F}_0 .

The cases $N = 2$ and $N = 3$ (with only one charge t) are given by

$$g_2(t, t, t) = \frac{1+t+13t^2+20t^3+53t^4+92t^5+137t^6+134t^7+146t^8+103t^9+55t^{10}+19t^{11}+9t^{12}}{(1-t)^8(1+t)^6(1+t+t^2)^3}$$

$$g_3(t, t, t) = \frac{1+32t^3+394t^6+2365t^9+7343t^{12}+12946t^{15}+13201t^{18}+7709t^{21}+2314t^{24}+276t^{27}+3t^{30}}{(1-t^3)^{11}(1+t^3)^3}$$

Taking the Plethystic Logarithm for these expressions we find 9 generators for $N = 1$, 18 baryonic and 10 mesonic for $N = 2$, 30 baryonic and 10 mesonic for $N = 3$, 45 baryonic, 10 mesonic of R charge 2, 28 mesonic of R charge 4 for $N = 4$, etc. By taking the order of the pole at $t = 1$ we find the dimension of the moduli space is $3N + 2$. All of this agrees with the field theory expectations.

5.3 Generating functions for \mathbb{F}_0

\mathbb{F}_0 , a \mathbb{Z}_2 freely acting orbifold of the conifold, has a quiver and the toric diagram given in figure 7. The quiver gauge theory has four types of fields **A**, **B**, **C**, **D**. The superpotential is $\epsilon_{ij}\epsilon_{pq}\mathbf{A}_i\mathbf{B}_p\mathbf{C}_j\mathbf{D}_q$.

Symmetries and geometry. Including flavor charges, we find a rank six global symmetry denoted by $SU(2)_1 \times SU(2)_2 \times U(1)_R \times U(1)_B \times U(1)_{A_1} \times U(1)_{A_2}$. The basic fields have transformation rules under the global symmetry which are summarized in table 3.

We can explicitly examine the geometry of \mathbb{F}_0 . To this purpose, since $d = 4$, we introduce four homogeneous coordinates x_i in \mathbb{C}^4 . As in section 4.2 we define the group of rescalings $\prod_{j=1}^4 \mu_j^{(e_i, V_j)} = 1$ which consists of a continuous charge acting as $(b, 1/b, b, 1/b)$ on the x_i and of a discrete one $(1, e, 1, e)$ with $e^2 = 1$. This implies that the manifold, as we know, is a \mathbb{Z}_2 quotient of the conifold

$$\mathcal{R}(\mathcal{C}) = \mathbb{C}[x_1, x_2, x_3, x_4]/(x_1x_2 - x_3x_4); \quad \mathcal{R}(\mathbb{F}_0) = \mathcal{R}(\mathcal{C})/\mathbb{Z}_2 \quad (5.24)$$

The homogeneous charges x_i can be represented by chemical potentials as

$$(x_1, x_2, x_3, x_4) \longrightarrow (t_1x, et_2y, t_1/x, et_2/y)$$

field	F_1	F_2	R	B	A_1	A_2	chemical potentials	non-anomalous chemical potentials
\mathbf{A}_1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	0	$tbxa_1$	$t_1x = tbx$
\mathbf{A}_2	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	1	0	$\frac{tba_1}{x}$	$\frac{t_1}{x} = \frac{tb}{x}$
\mathbf{B}_1	0	$\frac{1}{2}$	$\frac{1}{2}$	-1	0	1	$\frac{tya_2}{b}$	$t_2y = \frac{ty}{b}$
\mathbf{B}_2	0	$-\frac{1}{2}$	$\frac{1}{2}$	-1	0	1	$\frac{ta_2}{by}$	$\frac{t_2}{y} = \frac{t}{by}$
\mathbf{C}_1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	-1	0	$\frac{tbx}{a_1}$	$t_1x = tbx$
\mathbf{C}_2	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	-1	0	$\frac{tb}{xa_1}$	$\frac{t_1}{x} = \frac{tb}{x}$
\mathbf{D}_1	0	$\frac{1}{2}$	$\frac{1}{2}$	-1	0	-1	$\frac{ty}{ba_2}$	$t_2y = \frac{ty}{b}$
\mathbf{D}_2	0	$-\frac{1}{2}$	$\frac{1}{2}$	-1	0	-1	$\frac{t}{bya_2}$	$\frac{t_2}{y} = \frac{t}{by}$

Table 3: Global charges for the basic fields of the quiver gauge theory living on the D-brane probing the CY with \mathbb{F}_0 base.

in terms of the discrete baryonic charge e and the chemical potentials t_i and x, y ; notations are inherited from original conifold theory. For future reference, we notice that the right basis for localization is given by $q_i = \prod_{j=1}^4 x_j^{\langle e_i, V_j \rangle}$ and we compute $q_1 = x^2, q_2 = y^2, q_3 = t_1^2 t_2^2$.

5.3.1 The $N = 1$ generating function

The F terms of the theory read:

$$\begin{aligned} \mathbf{A}_1 \mathbf{B}_i \mathbf{C}_2 &= \mathbf{A}_2 \mathbf{B}_i \mathbf{C}_1, & \mathbf{B}_1 \mathbf{C}_i \mathbf{D}_2 &= \mathbf{B}_2 \mathbf{C}_i \mathbf{D}_1, \\ \mathbf{C}_1 \mathbf{D}_i \mathbf{A}_2 &= \mathbf{C}_2 \mathbf{D}_i \mathbf{A}_1, & \mathbf{D}_1 \mathbf{A}_i \mathbf{B}_2 &= \mathbf{D}_2 \mathbf{A}_i \mathbf{B}_1. \end{aligned}$$

For $N = 1$, the elementary fields are commuting numbers and in each equation a field factorizes. For example, the first equation reduces to $\mathbf{B}_i(\mathbf{A}_1 \mathbf{C}_2 - \mathbf{A}_2 \mathbf{C}_1) = 0$. This implies that the $N = 1$ moduli space is reducible to few different branches. At a generic point where all the fields are different from zero, we can divide by the common factor and the F-term equations reduce to

$$\mathbf{A}_1 \mathbf{C}_2 = \mathbf{A}_2 \mathbf{C}_1, \quad \mathbf{B}_1 \mathbf{D}_2 = \mathbf{B}_2 \mathbf{D}_1.$$

However, over the submanifold $\mathbf{B}_i = \mathbf{D}_i = 0$ the constraint $\mathbf{A}_1 \mathbf{C}_2 = \mathbf{A}_2 \mathbf{C}_1$ cannot be imposed and the dimension of the moduli space increases by one unit. The same applies for the constraint $\mathbf{B}_1 \mathbf{D}_2 = \mathbf{B}_2 \mathbf{D}_1$ over the submanifold $\mathbf{A}_i = \mathbf{C}_i = 0$.

This means that the moduli space is not irreducible and over particular submanifolds new branches are opening up. This is similar to what happens with the Coulomb branch of $\mathcal{N} = 2$ supersymmetric gauge theories.

We decide to study the irreducible components of the moduli space which contains the generic point with all fields different from zero. Algebraically, this is obtained by taking the closure of the open set $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \neq 0$. We will see that the geometry of the CY nicely captures this branch of the moduli space. The other branches can be added by performing surgeries as in [10]. The $N = 1$ generating function of the generic branch is given by the

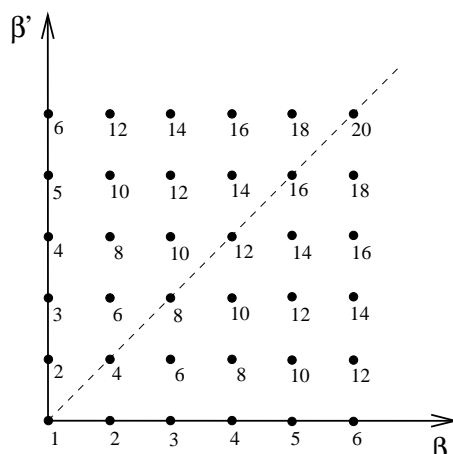


Figure 8: GKZ decomposition for \mathbb{F}_0 with multiplicities.

Hilbert series of the polynomial ring

$$\mathcal{R}_{N=1}(\mathbb{F}_0) = \mathbb{C}[\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i, \mathbf{D}_i] / (\mathbf{A}_1 \mathbf{C}_2 - \mathbf{A}_2 \mathbf{C}_1, \mathbf{B}_1 \mathbf{D}_2 - \mathbf{B}_2 \mathbf{D}_1) \quad (5.25)$$

In order to simplify expressions, we first set to 1 all chemical potentials in table 3 except for t_1 and t_2 . The Hilbert series is then easily computed by observing that equation (5.25) is a polynomial ring which is a complete intersection,

$$g_1(t_1, t_2; \mathbb{F}_0) = \frac{(1-t_1^2)(1-t_2^2)}{(1-t_1)^4(1-t_2)^4} \quad (5.26)$$

By taking order of the pole at $t_1 = t_2 = t = 1$ we find the dimension of the moduli space to be 6; this can be easily understood by having three mesonic directions (parameterizing the CY) plus three baryonic directions given by the gauge theory FI terms, consistent with a dimension formula of $3N + G - 1$.

5.3.2 The GKZ decomposition.

On the singular cone, there is just one independent divisor $D_1 = D_3 = -D_2 = -D_4$ as for the conifold. On the resolution, there is a new divisor D_5 corresponding to the internal point. $D_1 = D_3$ and $D_2 = D_4$ are still true but now D_1 and $-D_2$ are different. We can parametrize the GKZ fan in \mathbb{R}^2 with $\beta D_1 + \beta' D_2$. The integer parameters β and β' have the interpretation as the discrete Kähler parameters of \mathbb{F}_0 , namely the discrete areas of the two P^1 's. In this case, the convexity condition requires $\beta, \beta' \geq 0$. The GKZ cone is depicted in figure 8 and the multiplicities are presented in equation (5.32). Notice that the QFT baryonic charge is given by $B = \beta - \beta'$, so the sector $\beta' < \beta$ corresponds to the positive baryonic charges and the sector $\beta' > \beta$ corresponds to the negative ones.

It is interesting to note that the line $\beta' = \beta$ contains the sectors with zero baryonic charge. It is not however true that operators with zero baryonic charges are made with

traces; this is only true for $\beta = \beta' = 0$. The other sectors on the line $\beta' = \beta$ correspond to determinants of the form $(\det \mathbf{A} \det \mathbf{B})^n$.

Localization now gives (using the complete set of charges $(t_1 x, t_2 y, t_1/x, t_2/y)$)

$$g_{1,\beta,\beta'}(t_1, t_2, x, y; \mathbb{F}_0) = \frac{t_1^\beta t_2^{\beta'} x^{-\beta} y^{-\beta'}}{(1-x^2) \left(1 - \frac{t_1^2 t_2^2}{x^2 y^2}\right) (1-y^2)} + \frac{t_1^\beta t_2^{\beta'} x^\beta y^{-\beta'}}{(1-1/x^2) (1-t_1^2 t_2^2 x^2/y^2) (1-y^2)} \quad (5.27)$$

$$+ \frac{t_1^\beta t_2^{\beta'} x^{-\beta} y^{\beta'}}{(1-x^2) (1-t_1^2 t_2^2 y^2/x^2) (1-1/y^2)} + \frac{t_1^\beta t_2^{\beta'} x^\beta y^{\beta'}}{(1-1/x^2) (1-t_1^2 t_2^2 x^2 y^2) (1-1/y^2)}$$

The dependence on the baryonic charge can be obtained by replacing $t_1 \rightarrow tb$ and $t_2 \rightarrow t/b$ and, as expected, is given by a multiplicative factor

$$g_{1,\beta,\beta'}(t_1, t_2, x, y; \mathbb{F}_0) = b^{\beta-\beta'} \hat{g}_{1,\beta,\beta'}(t, x, y; \mathbb{F}_0).$$

The generating function for $x, y = 1$ can be nicely written as

$$g_{1,\beta,\beta'}(t_1, t_2; \mathbb{F}_0) = \sum_{n=0}^{\infty} (2n+1+\beta)(2n+1+\beta') t_1^{2n+\beta} t_2^{2n+\beta'} \quad (5.28)$$

It is then obvious that, for example, the mesonic partition function $g_{1,0,0}$ can be obtained from the mesonic partition function for the conifold [50, 6] by projecting on the \mathbb{Z}_2 invariant part ($t_2 \rightarrow -t_2$).

5.3.3 Multiplicities

To extract multiplicities we use the auxiliary partition function for the GKZ cone. We introduce letters a, b, c, d for the four possible classes of arrows. The only relation that they form is related to the closed loop $abcd$. The generating function is the Hilbert series of the polynomial ring

$$\mathcal{R}_{\text{GKZ}}(\mathbb{F}_0) = \mathbb{C}[a, b, c, d]/(abcd) \quad (5.29)$$

By assigning chemical potential t_1 to a, c and t_2 to b, d we obtain the auxiliary GKZ partition function for multiplicities:

$$Z_{\text{aux}}(t_1, t_2; \mathbb{F}_0) = \frac{1 - t_1^2 t_2^2}{(1-t_1)^2 (1-t_2)^2} \quad (5.30)$$

$$= 1 + \sum_{\beta=1}^{\infty} (\beta+1) t_1^\beta + \sum_{\beta'=1}^{\infty} (\beta'+1) t_2^{\beta'} + \sum_{\beta=1}^{\infty} \sum_{\beta'=1}^{\infty} 2(\beta+\beta') t_1^\beta t_2^{\beta'}$$

From which we can extract the following multiplicities, $2(\beta+\beta')$ for $\beta, \beta' \geq 1$, $\beta+1$ for $\beta' = 0$, and $\beta'+1$ for $\beta = 0$.

We thus have

$$g_1(\{t_i\}; \mathbb{F}_0) = g_{1,0,0} + \sum_{\beta=1}^{\infty} (\beta+1) g_{1,\beta,0} + \sum_{\beta'=1}^{\infty} (\beta'+1) g_{1,0,\beta'} + \sum_{\beta,\beta'=1}^{\infty} 2(\beta+\beta') g_{1,\beta,\beta'} \quad (5.31)$$

and one computes, using equation (5.28),

$$g_1(t_1, t_2; \mathbb{F}_0) = \frac{(1-t_1^2)(1-t_2^2)}{(1-t_1)^4 (1-t_2)^4}$$

which is exactly equation (5.26).

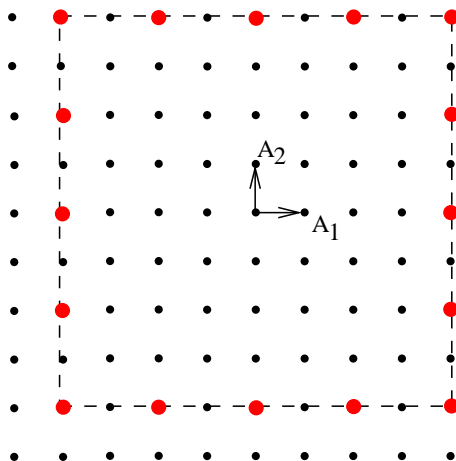


Figure 9: The hollow rectangle $C(4, 4)$ above $(\beta, \beta') = (4, 4)$. It gives the multiplicity 16.

5.3.4 Refining the GKZ decomposition

The auxiliary partition function in equation (5.30) is derived by computing the Hilbert series of the GKZ ring in equation (5.29). By expanding this partition function we get multiplicities in the (β, β') lattice given by the following infinite matrix,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & & \\ 2 & 4 & 6 & 8 & 10 & 12 & & \\ 3 & 6 & 8 & 10 & 12 & 14 & \dots & \\ 4 & 8 & 10 & 12 & 14 & 16 & & \\ 5 & 10 & 12 & 14 & 16 & 18 & & \\ 6 & 12 & 14 & 16 & 18 & 20 & & \\ & \vdots & & & & & \ddots & \end{pmatrix}. \quad (5.32)$$

In order to avoid getting multiplicities, let us introduce the chemical potentials for the anomalous charges given in table 3. Using Macaulay2, or the fact that we are dealing with a complete intersection, we can write the auxiliary partition function dressed with these new charges as

$$Z_{\text{aux}}(t_1, t_2, a_1, a_2; \mathbb{F}_0) = \frac{1 - t_1^2 t_2^2}{(1 - t_1 a_1)(1 - t_1/a_1)(1 - t_2 a_2)(1 - t_2/a_2)} \quad (5.33)$$

By expanding this function, we see that above each point in the GKZ fan, parametrized by (β, β') , there is a rectangle $C(\beta, \beta')$ in the (A_1, A_2) lattice as in figure 9. The related rectangle in the pq-web is shown in figure 10.

We can thus refine our decomposition of the $N = 1$ partition function. Equation (5.31) can be replaced by the following formula where all multiplicities are lifted:

$$g_1(t_1, t_2, a_1, a_2) = \sum_{\beta=0, \beta'=0}^{\infty} \left(\sum_{K \in C(\beta, \beta')} a_1^{K_1} a_2^{K_2} \right) g_{1, \beta, \beta'}(t_1, t_2). \quad (5.34)$$

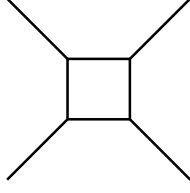


Figure 10: The pq-web of \mathbb{F}_0 . The blown-up four-cycle is associated to the square in the middle.

One can explicitly resum the right hand side, computed from geometrical data and the auxiliary GKZ generating functions, and compare it with the $N = 1$ generating function as computed by field theory. By resumming the series in equation (5.34) we indeed recover the the generating function for $N = 1$ with all anomalous and non-anomalous charges; this is given by the Hilbert series for the polynomial ring of equation (5.25) and is easily computed using the fact that we deal with a complete intersection,

$$g_1(x, y, t, b, a_1, a_2; \mathbb{F}_0) = \frac{(1 - \frac{t^2}{b^2})(1 - b^2 t^2)}{\left(1 - \frac{bt}{a_1 x}\right) \left(1 - \frac{a_1 bt}{x}\right) \left(1 - \frac{btx}{a_1}\right) (1 - a_1 btx) \left(1 - \frac{t}{a_2 by}\right) \left(1 - \frac{a_2 t}{by}\right) \left(1 - \frac{ty}{a_2 b}\right) \left(1 - \frac{a_2 ty}{b}\right)} \quad (5.35)$$

For completeness we can rewrite this expression by summing using equations (5.27) and (5.33),

$$g_1(x, y, t, b, a_1, a_2; \mathbb{F}_0) = \frac{1}{(1-x^2)(1-t_1 a_1/x) \left(1 - \frac{t_1}{x a_1}\right) (1-t_2 a_2/y) \left(1 - \frac{t_2}{y a_2}\right) (1-y^2)} + \frac{1}{(1-1/x^2)(1-t_1 x a_1)(1-t_1 x/a_1)(1-t_2 a_2/y)(1-t_2/y a_2)(1-y^2)} + \frac{1}{(1-x^2)(1-t_1 a_1/x)(1-t_1/x a_1)(1-t_2 y a_2)(1-t_2 y/a_2)(1-1/y^2)} + \frac{1}{\left(1 - \frac{1}{x^2}\right) (1-t_1 x a_1)(1-t_1 x/a_1)(1-t_2 y a_2)(1-t_2 y/a_2)(1-1/y^2)}. \quad (5.36)$$

This formula suggests that some localization is at work in the field theory $N = 1$ moduli space, which is a six dimensional variety with the action of six $U(1)$ flavor and baryonic symmetries.

5.3.5 Expansion in baryonic charges

Equation (5.26) can be refined by including the two anomalous chemical potentials a_1 and a_2 ,

$$g_1(t_1, t_2, a_1, a_2; \mathbb{F}_0) = \frac{(1-t_1^2)(1-t_2^2)}{(1-t_1 a_1)^2 \left(1 - \frac{t_1}{a_1}\right)^2 (1-t_2 a_2)^2 \left(1 - \frac{t_2}{a_2}\right)^2} \quad (5.37)$$

and by using residue formulae we can expand in terms of generating functions with fixed anomalous baryonic charges,

$$g_{1,A_1,A_2}(t_1, t_2; \mathbb{F}_0) = \frac{t_1^{|A_1|} t_2^{|A_2|} [1 + t_1^2 + |A_1|(1 - t_1^2)][1 + t_2^2 + |A_2|(1 - t_2^2)]}{(1 - t_1^2)^2 (1 - t_2^2)^2} \quad (5.38)$$

More generally, we can make explicit the dependence of the $N = 1$ generating function on the full set of baryonic charges

$$g_1(t, b, a_1, a_2; \mathbb{F}_0) = \frac{(1 - t^2 b^2) \left(1 - \frac{t^2}{b^2}\right)}{(1 - t b a_1)^2 \left(1 - \frac{t b}{a_1}\right)^2 (1 - \frac{t a_2}{b})^2 \left(1 - \frac{t}{b a_2}\right)^2} \quad (5.39)$$

which can be expanded in sectors with definite baryonic charges. A contour integral argument gives the following generating functions for 1 D-brane and for fixed baryonic charges

$$g_{1,B,A_1,A_2}(t; \mathbb{F}_0) = \frac{1 + (-1)^{B+A_1+A_2}}{2(1 - t^4)^3} \begin{cases} t^{-B-2A_1} [1 + 6t^4 + t^8 - (B + 2A_1)(1 - t^8) + \\ + (A_1^2 + B A_1)(1 - t^4)^2] & A_1 \leq 0, \quad |A_2| \leq -B - A_1 \\ t^{-B+2A_1} [1 + 6t^4 + t^8 - (B - 2A_1)(1 - t^8) + \\ + (A_1^2 - B A_1)(1 - t^4)^2] & A_1 \geq 0, \quad |A_2| \leq -B + A_1 \\ t^{B-2A_2} [1 + 6t^4 + t^8 + (B - 2A_2)(1 - t^8) + \\ + (A_2^2 - B A_2)(1 - t^4)^2] & A_2 \leq 0, \quad |A_1| \leq B - A_2 \\ t^{B+2A_2} [1 + 6t^4 + t^8 + (B + 2A_2)(1 - t^8) + \\ + (A_2^2 + B A_2)(1 - t^4)^2] & A_2 \geq 0, \quad |A_1| \leq B + A_2 \end{cases} \quad (5.40)$$

The same result can be obtained by projecting the decomposition of the $N = 1$ generating function on the refined GKZ lattice on the three dimensional space containing the three baryonic charges. This is done by projecting the two GKZ coordinates (β, β') to the baryonic charge $B = \beta - \beta'$ and keeping fixed the two anomalous charges A_1 and A_2 , restricted to the conditions $-\beta \leq A_1 \leq \beta$ and $-\beta' \leq A_2 \leq \beta'$.

5.3.6 Generating functions for $N > 1$

As before the higher N generating function g_N is given by the plethystic exponentiation using formula (2.2). We can start from *any* decomposition of the $N = 1$ generating function, the GKZ decomposition (5.31), the refined GKZ decomposition (5.34) or the expansion in baryonic charges (5.40). Using the GKZ decomposition we have

$$\begin{aligned} \sum_{N=0}^{\infty} g_N(\{t_i\}; \mathbb{F}_0) \nu^N &= \text{PE}_\nu[g_{1,0,0}] + \sum_{\beta=1}^{\infty} (\beta + 1) \text{PE}_\nu[g_{1,\beta,0}] \\ &+ \sum_{\beta'=1}^{\infty} (\beta' + 1) \text{PE}_\nu[g_{1,0,\beta'}] + \sum_{\beta=1}^{\infty} \sum_{\beta'=1}^{\infty} 2(\beta + \beta') \text{PE}_\nu[g_{1,\beta,\beta'}] \end{aligned} \quad (5.41)$$

From which we can compute the generating function for $N = 2$

$$g_2(t_1, t_2) = \frac{F(t_1, t_2)}{(1 - t_1^2)^4(1 - t_1^2 t_2^2)^3(1 - t_2^2)^4}$$

$$F(t_1, t_2) = 1 + 2t_2^2 + t_1^8 t_2^4(-6 + 8t_2^2 + 17t_2^4 + 2t_2^6) + t_1^2(2 + 14t_2^2 + 8t_2^4 - 3t_2^6) - t_1^6 t_2^2(3 + 20t_2^2 + 10t_2^4 - 8t_2^6 - t_2^8) + t_1^{10}(t_2^6 + 2t_2^8) - t_1^4(-8t_2^2 + 6t_2^4 + 20t_2^6 + 6t_2^8)$$

Taking the plethystic logarithm of the generating functions for \mathbb{F}_0 , it is possible to find the generators of the moduli space. For $N = 1$ the plethystic logarithm of (5.26) is very simple:

$$(4t_1 + 4t_2) - (t_1^2 + t_2^2)$$

and it correctly reproduces the 8 chiral field generators and the two relations among them. For $N = 1$ the moduli space is a complete intersection, and this is reflected by the fact that the plethystic logarithm has a finite number of terms. In the $N = 2$ case there are 12 baryonic generators and 13 mesonic ones and the moduli space is no more a complete intersection. By looking at the order of the pole for the computed generating functions near $t_1 = t_2 = t = 1$, we checked that the dimension of the moduli space is $3N + G - 1$ as expected.

5.4 Counting in $\frac{1}{2}\mathbb{F}_0$ and in $\frac{3}{4}\mathbb{F}_0$

We can also consider, as was done for the conifold in [12], fictitious theories counting subsets of the BPS chiral operators.

The half \mathbb{F}_0 corresponds to considering only the BPS operators containing $\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1, \mathbf{D}_1$ and no occurrences of the other four elementary fields. This is done by truncating equation (5.28) for $g_{1,\beta,\beta'}$

$$g_{1,\beta,\beta'} = \sum_{n=0}^{\infty} t_1^{2n+\beta} t_2^{2n+\beta'} \tag{5.42}$$

We easily obtain the generating functions for $N = 1, 2$,

$$g_1\left(t_1, t_2; \frac{1}{2}\mathbb{F}_0\right) = \frac{1}{(1 - t_1)^2(1 - t_2)^2}$$

$$g_2\left(t_1, t_2; \frac{1}{2}\mathbb{F}_0\right) = \frac{1}{(1 - t_1^2)^2(1 - t_1^2 t_2^2)(1 - t_2^2)^2} \tag{5.43}$$

Moreover, by generalizing arguments given in [12] for the 1/2 conifold, it is easy to write a formula for generic N

$$g_N\left(t_1, t_2; \frac{1}{2}\mathbb{F}_0\right) = \frac{1}{(1 - t_1^N)^2(1 - t_2^N)^2} \prod_{i=1}^{N-1} \frac{1}{1 - t_1^{2i} t_2^{2i}} \tag{5.44}$$

which is interpreted as the fact that the ring of invariants is freely generated by four determinants $\det \mathbf{A}_1, \det \mathbf{B}_1, \det \mathbf{C}_1, \det \mathbf{D}_1$ and $N - 1$ mesons $\text{Tr}(\mathbf{A}_1 \mathbf{B}_1 \mathbf{C}_1 \mathbf{D}_1)^i$ with $i = 1, \dots, N - 1$. This can be understood easily from the absence of nontrivial F term conditions and the fact that all baryons factorize into an elementary determinant times mesons.

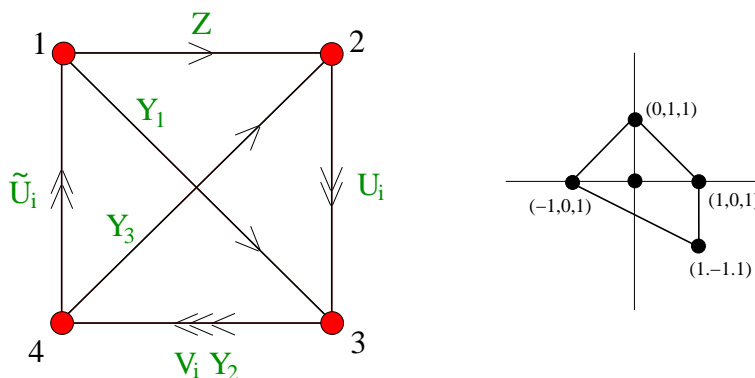


Figure 11: Quiver and toric diagram for dP_1 .

The $\frac{3}{4}\mathbb{F}_0$ corresponds to considering only the BPS operators containing $\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1, \mathbf{D}_1$ and $\mathbf{A}_2, \mathbf{C}_2$ and no occurrences of the other two elementary fields. This is done by taking

$$g_{1,\beta,\beta'} = \sum_{n=0}^{\infty} (2n + 1 + \beta) t_1^{2n+\beta} t_2^{2n+\beta'}. \quad (5.45)$$

Using the same multiplicities as in equation (5.31) we obtain

$$\begin{aligned} g_1 \left(t_1, t_2; \frac{3}{4}\mathbb{F}_0 \right) &= \frac{1 + t_1}{(1 - t_1)^3 (1 - t_2)^2} \\ g_2 \left(t_1, t_2; \frac{3}{4}\mathbb{F}_0 \right) &= \frac{(1 + t_1^2 t_2^2)(1 + 2t_1^2 + 2t_1^4 t_2^2 + t_1^6 t_2^2)}{(1 - t_1^2)^4 (1 - t_1^2 t_2^2)^2 (1 - t_2^2)^2} \end{aligned} \quad (5.46)$$

5.5 Generating functions for del Pezzo 1

The quiver and the toric diagram are depicted in figures 11 and 12.

Including flavor charges, we find a rank six global symmetry denoted by $SU(2) \times U(1)_F \times U(1)_{R_0} \times U(1)_B \times U(1)_{A_1} \times U(1)_{A_2}$. R_0 is not the R -symmetry of the superconformal theory but rather a more convenient parametrization of the fields for the purpose of the counting problem. It counts the fields in way such that if the dP_1 theory is Higgsed down to the $\mathbb{C}^3/\mathbb{Z}_3$ theory by giving a vev to the \mathbf{Z} field, then R_0 measures the R -symmetry of this resulting theory. The basic fields have transformation rules under the global symmetry which are summarized in table 4.

5.5.1 The $N = 1$ generating function

The polynomial ring for the $N = 1$ moduli space is

$$\mathcal{R}_{N=1}(dP_1) = \mathbb{C}[\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Z}, \mathbf{U}_1, \mathbf{U}_2, \tilde{\mathbf{U}}_1, \tilde{\mathbf{U}}_2, \mathbf{V}_1, \mathbf{V}_2] / \{\partial W = 0\} \quad (5.47)$$

where $W = \epsilon_{ab} \mathbf{Y}_1 \mathbf{V}_a \tilde{\mathbf{U}}_b + \epsilon_{ab} \mathbf{Y}_3 \mathbf{U}_a \mathbf{V}_b + \epsilon_{ab} \mathbf{Y}_2 \tilde{\mathbf{U}}_a \mathbf{Z} \mathbf{U}_b$.

field	SU(2)	F	R_0	B	A_1	A_2	chemical potentials	non-anomalous ch. potentials	GKZ letters
\mathbf{U}_1	$\frac{1}{2}$	0	$\frac{2}{3}$	-2	1	0	$\frac{tx}{b^2}a_1$	$x_2 = \frac{tx}{b^2}$	d
\mathbf{U}_2	$-\frac{1}{2}$	0	$\frac{2}{3}$	-2	1	0	$\frac{t}{xb^2}a_1$	$x_4 = \frac{t}{xb^2}$	d
$\tilde{\mathbf{U}}_1$	$\frac{1}{2}$	0	$\frac{2}{3}$	-2	-1	0	$\frac{tx}{b^2}a_1$	$x_2 = \frac{tx}{b^2}$	b
$\tilde{\mathbf{U}}_2$	$-\frac{1}{2}$	0	$\frac{2}{3}$	-2	-1	0	$\frac{t}{xb^2}a_1$	$x_4 = \frac{t}{xb^2}$	b
\mathbf{V}_1	$\frac{1}{2}$	-1	$\frac{2}{3}$	1	0	-1	$\frac{tbx}{ya_2}$	$x_1x_2 = \frac{tbx}{y}$	a
\mathbf{V}_2	$-\frac{1}{2}$	-1	$\frac{2}{3}$	1	0	-1	$\frac{tb}{xya_2}$	$x_1x_4 = \frac{tb}{xy}$	a
\mathbf{Y}_1	0	1	$\frac{2}{3}$	1	1	1	$tbya_1a_2$	$x_3 = tby$	f
\mathbf{Y}_2	0	1	$\frac{2}{3}$	1	0	-1	$\frac{tby}{a_2}$	$x_3 = tby$	a
\mathbf{Y}_3	0	1	$\frac{2}{3}$	1	-1	1	$\frac{tbya_2}{a_1}$	$x_3 = tby$	e
\mathbf{Z}	0	-1	0	3	0	1	$\frac{b^3a_2}{y}$	$x_1 = \frac{b^3}{y}$	c

Table 4: Global charges for the basic fields of the quiver gauge theory living on the D-brane probing the CY with dP_1 base.

There are 10 different F-term equations. Consider in particular the equations for \mathbf{Z} and \mathbf{Y}_2

$$\epsilon_{ab}\tilde{\mathbf{U}}_a\mathbf{Z}\mathbf{U}_b = 0 \quad \epsilon_{ab}\mathbf{U}_b\mathbf{Y}_2\tilde{\mathbf{U}}_a = 0 \quad (5.48)$$

The fact that we can factorize a field in each of these equations implies that the moduli space of vacua is not irreducible. Over the submanifold $\mathbf{Z} = \mathbf{Y}_2 = 0$ the dimension of the moduli space increases by one unit.

Instead of the two conditions (5.48) we will impose the simpler condition

$$\epsilon_{ab}\tilde{\mathbf{U}}_a\mathbf{U}_b = 0. \quad (5.49)$$

As in the \mathbb{F}_0 example, this means that we are considering one irreducible component of the moduli space, which is the closure of the open set $\mathbf{Z}, \mathbf{Y}_2 \neq 0$. This is the branch that is nicely described by the CY geometry.

We give chemical potentials to the fields using four homogeneous coordinates x_i as in table 4 as described in [35, 55]. With these weights we can compute the Hilbert series for the graded ring, equation (5.47), by using Macaulay2 and obtain the $N = 1$ generating function $g_1(x_i)$,

$$g_1(\{x_i\}; dP_1) = \frac{Q(x_i)}{(1-x_1)(1-x_2)^2(1-x_1x_2)(1-x_3)^3(1-x_4)^2(1-x_1x_4)} \quad (5.50)$$

where

$$\begin{aligned} Q(x_i) = & 1 - x_2x_3 - 2x_1x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 - x_2x_4 - 2x_1x_2x_4 + x_1x_2^2x_4 - x_3x_4 \\ & - 2x_1x_3x_4 + 2x_2x_3x_4 + 6x_1x_2x_3x_4 + 2x_1^2x_2x_3x_4 - 2x_1x_2^2x_3x_4 - x_1^2x_2^2x_3x_4 + x_1x_3^2x_4 \\ & - 2x_1x_2x_3^2x_4 - x_1^2x_2x_3^2x_4 + x_1x_2x_4^2 + x_1x_3x_4^2 - 2x_1x_2x_3x_4^2 - x_1^2x_2x_3x_4^2 + x_1^2x_2^2x_3^2x_4^2 \end{aligned}$$

Looking at the order of the pole at $x_1 = x_2 = x_3 = x_4 \rightarrow 1$ we find that the dimension of the moduli space is 6, in agreement with our general formula $3N + G - 1$.

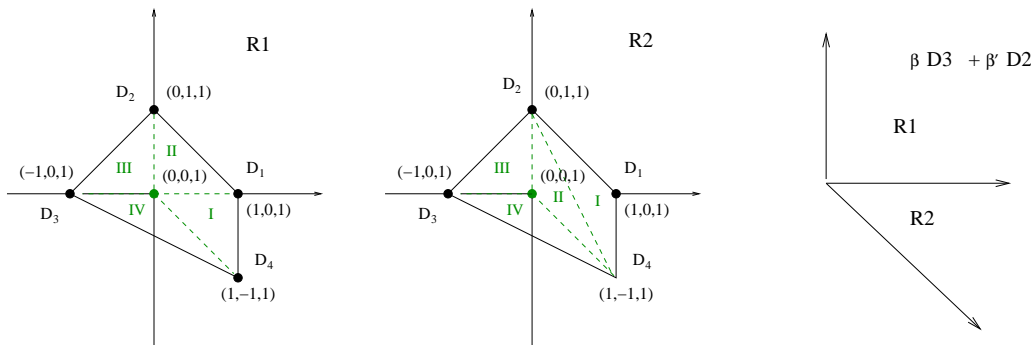


Figure 12: The two resolutions, the quiver and GKZ decomposition for dP_1 .

5.5.2 The GKZ decomposition.

The GKZ fan for dP_1 is depicted in figure 12. Including the internal points we have five divisors D_i subject to the equivalence relations $D_2 = D_4$, $D_5 = -D_2 - 2D_3$, $D_1 = D_3 - D_2$. This leaves a $I + d - 3 = 2$ -dimensional space of Kähler parameters, which we can identify with the plane D_2, D_3 . We have

$$\sum_{i=1}^5 a_i D_i = (a_3 + a_1 - 2a_5)D_3 + (a_2 - a_1 + a_4 - a_5)D_2 \equiv \beta D_3 + \beta' D_2 \quad (5.51)$$

Even if $a_i \geq 0$, β and β' can be negative. There are two smooth resolutions $R1$ and $R2$. Assign the number β to the vertex D_3 and β' to the vertex D_2 of the toric diagram, and zero to all other vertices. The convexity conditions for $R1$ give $\beta, \beta' \geq 0$ and the conditions for $R2$ give $\beta \geq 0, \beta' \leq 0, \beta + \beta' \geq 0$. These two sets of conditions determine the two adjacent cones in the GKZ decomposition. Notice that the three boundary lines can be associated naturally with the directions D_2, D_3 and $D_1 = D_3 - D_2$. We go from $R1$ to $R2$ with a flop.

Collecting this together and using equation (4.8) we obtain

$$\begin{aligned} g_{1,\beta,\beta'}^{R1}(x_i) &= \frac{x_2^{\beta'} x_3^\beta}{(1-x_1 x_2/x_3)(1-x_2 x_3^2)(1-x_4/x_2)} + \frac{x_3^\beta x_4^{\beta'}}{(1-x_2/x_4)(1-x_1 x_4/x_3)(1-x_3^2 x_4)} \\ &+ \frac{x_4^{\beta+\beta'} x_1^\beta}{(1-x_2/x_4)(1-x_3/x_1 x_4)(1-x_4^3 x_1^2)} + \frac{x_1^\beta x_2^{\beta+\beta'}}{(1-x_1^2 x_2^3)(1-x_3/x_1 x_2)(1-x_4/x_2)} \\ g_{1,\beta,\beta'}^{R2}(x_i) &= \frac{x_3^{\beta-2\beta'}}{(1-x_1/x_3^3)(1-x_2 x_3^2)(1-x_3^2 x_4)} + \frac{x_1^{-\beta'} x_3^{\beta+\beta'}}{(1-x_1 x_2/x_3)(1-x_3^3/x_1)(1-x_1 x_4/x_3)} \\ &+ \frac{x_1^\beta x_4^{\beta+\beta'}}{(1-x_2/x_4)(1-x_3/x_1 x_4)(1-x_4^3 x_1^2)} + \frac{x_1^\beta x_2^{\beta+\beta'}}{(1-x_1^2 x_2^3)(1-x_3/x_1 x_2)(1-x_4/x_2)} \end{aligned} \quad (5.52)$$

5.5.3 Multiplicities

We determine multiplicities using the auxiliary partition functions for the GKZ cone. We have six different equivalence classes of fields a, b, c, d, e, f corresponding to the fields as in

table 4. The relations are

$$\mathcal{I} = \{abcd, afd, abe, cd, bc, eaf, fd - be\}$$

The first three relations correspond to closed loops in the quiver, the composition of arrows cd , bc and eaf are equivalent to the e, f and c respectively and should be set to zero in order to avoid overcounting and, finally, fd and be are identified since they have the same starting and ending point. Not all the relations are independent.

We can assign charges to the letters a, b, c, \dots by considering their representative in terms of elementary fields and using the homogeneous charge assignment given in table 4. In the GKZ plane we want to use the charges x_2 and x_3 associated with D_2 and D_3 . The restriction to the GKZ plane requires using the equivalence relations $D_4 = D_2$ and $D_1 = D_3 - D_2$ that translate to the restriction to $x_4 = x_2$ and $x_1 = x_3/x_2$. We thus obtain the assignment

$$\{a, b, c, d, e, f\} \longrightarrow \{x_3, x_2, x_3/x_2, x_2, x_3, x_3\}. \quad (5.53)$$

Using Macaulay2 to compute the Hilbert series for the polynomial ring

$$\mathcal{R}_{\text{GKZ}}(dP_1) = \mathbb{C}[a, b, c, d, e, f]/\mathcal{I} \quad (5.54)$$

we obtain the auxiliary GKZ partition function⁷

$$Z_{\text{aux}}(x_2, x_3; dP_1) = \frac{1 - x_3^2 x_2}{(1 - x_3)(1 - x_3/x_2)(1 - x_2)^2} = \sum_{\beta, \beta'} m(\beta, \beta') x_3^\beta x_2^{\beta'} \quad (5.55)$$

which is expanded in power series for $x_3 < 1, x_2 < 1, x_3/x_2 < 1$. It exactly fills the regions R1 and R2 of the GKZ fan. We can also extract the multiplicities: in the internal points of region R1 $m(\beta, \beta') = 3\beta + 2\beta'$, in the internal points of region R2 $m(\beta, \beta') = 3(\beta + \beta')$; at the origin $m(0, 0) = 1$, on the vertical axis $m(0, \beta') = \beta' + 1$, on the horizontal axis $m(\beta, 0) = 3\beta$ and finally on the diagonal $m(\beta, -\beta) = 1$.

The $N = 1$ generating function is a sum over the two GKZ regions:

$$g_1(\{x_i\}) = \sum_{\beta, \beta' \in R1} m(\beta, \beta') g_{1, \beta, \beta'}^{R1}(x_i) + \sum_{\beta, \beta' \in R2} m(\beta, \beta') g_{1, \beta, \beta'}^{R2}(x_i) \quad (5.56)$$

By resumming this formula, we obtain precisely equation (5.50).

5.5.4 Refinement of the GKZ Lattice

Now we add the anomalous charges according to table 4. Macaulay2 computes the GKZ partition function to be

$$Z_{\text{aux}}(x_2, x_3, a_1, a_2; dP_1) = \frac{1 - x_2 x_3^2}{\left(1 - \frac{x_3 a_2}{x_2}\right) (1 - x_2 a_1) \left(1 - \frac{x_2}{a_1}\right) \left(1 - \frac{x_3}{a_2}\right)} \quad (5.57)$$

⁷The auxiliary polynomial ring of equation (5.54) has in fact the same Hilbert series as for a simpler polynomial ring $\mathbb{C}[a, b, c, d]/\{abcd\}$.

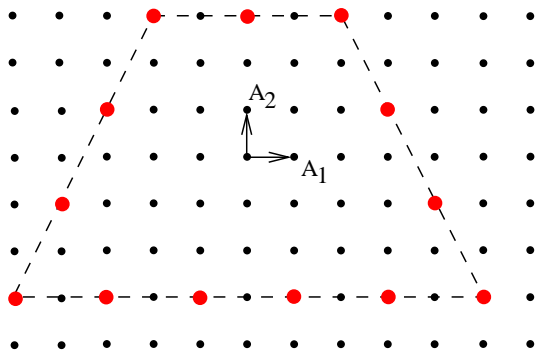


Figure 13: The hollow trapezoid above $(\beta, \beta') = (4, 3)$ which is in region R1. It gives the multiplicity 13.

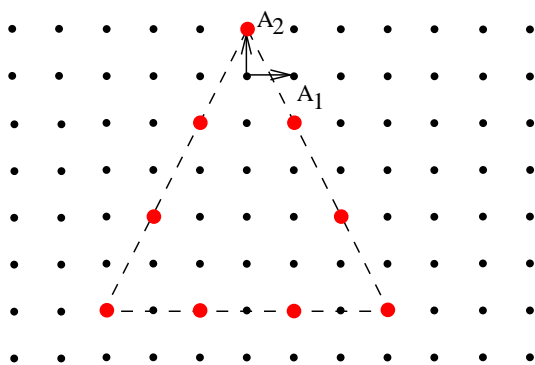


Figure 14: The hollow triangle above $(\beta, \beta') = (6, -1)$ which is in region R2. It gives the multiplicity 9.

The expansion gives a hollow cone in the four dimensional (B, B', A_1, A_2) lattice. In the R1 region, we have a trapezoid $C_{R1}(\beta, \beta')$ in the (A_1, A_2) lattice above a point in the GKZ cone parameterized by (β, β') (see figure 13). This degenerates to a triangle $C_{R2}(\beta, \beta')$ as we move to the R2 region (figure 14). This “explains” the $m(\beta, \beta') = 2\beta' + 3\beta$ multiplicities.

The $N = 1$ decomposition in equation (5.56) can be refined to

$$\begin{aligned}
 g_1(x_i, a_1, a_2; dP_1) = & \sum_{\beta, \beta' \in R1} \left(\sum_{K \in C_{R1}(\beta, \beta')} a_1^{K_1} a_2^{K_2} \right) g_{1, \beta, \beta'}^{R1}(x_i) \\
 & + \sum_{\beta, \beta' \in R2} \left(\sum_{K \in C_{R2}(\beta, \beta')} a_1^{K_1} a_2^{K_2} \right) g_{1, \beta, \beta'}^{R2}(x_i)
 \end{aligned} \tag{5.58}$$

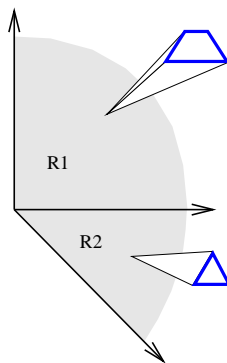


Figure 15: The two cones (grey areas) of the dP_1 GKZ fan. Above the points in $R1$ a trapezoid sits in the fiber. The edge lengths are controlled by the position in the base. The trapezoid degenerates to a triangle in the $R2$ region.

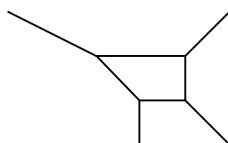


Figure 16: The pq-web of dP_1 . The blown-up four-cycle is associated to the trapezoid in the middle.

where the generating function for $N = 1$ depending on all charges is:

$$g_1(x_1, x_2, x_3, x_4, a_1, a_2; dP_1) = \frac{Q(x_1, x_2, x_3, x_4, a_1, a_2)}{(1-a_2x_1)\left(1-\frac{x_2}{a_1}\right)(1-a_1x_2)\left(1-\frac{x_1x_2}{a_2}\right)\left(1-\frac{x_3}{a_2}\right)\left(1-\frac{a_2x_3}{a_1}\right)(1-a_1a_2x_3)\left(1-\frac{x_4}{a_1}\right)(1-a_1x_4)\left(1-\frac{x_1x_4}{a_2}\right)} \quad (5.59)$$

where $Q(x_1, x_2, x_3, x_4, a_1, a_2)$ is the polynomial:

$$\begin{aligned} Q(x_1, x_2, x_3, x_4, a_1, a_2) = & 1 - a_2x_2x_3 - \frac{x_1x_2x_3}{a_1} - a_1x_1x_2x_3 + x_1x_2^2x_3 + a_2x_1x_2x_3^2 - x_2x_4 \\ & - \frac{x_1x_2x_4}{a_1a_2} - \frac{a_1x_1x_2x_4}{a_2} + \frac{x_1x_2^2x_4}{a_2} - a_2x_3x_4 - \frac{x_1x_3x_4}{a_1} - a_1x_1x_3x_4 \\ & + \frac{a_2x_2x_3x_4}{a_1} + a_1a_2x_2x_3x_4 + 4x_1x_2x_3x_4 + \frac{x_1x_2x_3x_4}{a_1^2} + a_1^2x_1x_2x_3x_4 \\ & + \frac{x_1^2x_2x_3x_4}{a_1a_2} + \frac{a_1x_1^2x_2x_3x_4}{a_2} - \frac{x_1x_2^2x_3x_4}{a_1} - a_1x_1x_2^2x_3x_4 - \frac{x_1^2x_2^2x_3x_4}{a_2} \\ & + a_2x_1x_3^2x_4 - \frac{a_2x_1x_2x_3^2x_4}{a_1} - a_1a_2x_1x_2x_3^2x_4 - x_1^2x_2x_3^2x_4 + \frac{x_1x_2x_4^2}{a_2} \\ & + x_1x_3x_4^2 - \frac{x_1x_2x_3x_4^2}{a_1} - a_1x_1x_2x_3x_4^2 - \frac{x_1^2x_2x_3x_4^2}{a_2} + x_1^2x_2^2x_3^2x_4^2 \quad (5.60) \end{aligned}$$

This expression can be rewritten in a more symmetric form by using equations (5.52) and (5.57) and some algebraic manipulation,

$$\begin{aligned}
 g_1(x_1, x_2, x_3, x_4, a_1, a_2; dP_1) = & \frac{1}{(1-x_1a_2)(1-x_2a_1)\left(1-\frac{x_2}{a_1}\right)\left(1-\frac{x_1x_2}{a_2}\right)\left(1-\frac{x_3}{x_1x_2}\right)\left(1-\frac{x_4}{x_2}\right)} \\
 & + \frac{1}{\left(1-\frac{x_1x_2}{x_3}\right)\left(1-\frac{x_3a_2}{x_2}\right)(1-x_2a_1)\left(1-\frac{x_2}{a_1}\right)\left(1-\frac{x_3}{a_2}\right)\left(1-\frac{x_4}{x_2}\right)} \\
 & + \frac{1}{\left(1-\frac{x_2}{x_4}\right)\left(1-\frac{x_1x_4}{x_3}\right)\left(1-\frac{x_3a_2}{x_4}\right)(1-x_4a_1)\left(1-\frac{x_4}{a_1}\right)\left(1-\frac{x_3}{a_2}\right)} \\
 & + \frac{1}{\left(1-\frac{x_2}{x_4}\right)\left(1-\frac{x_3}{x_1x_4}\right)(1-x_1a_2)(1-x_4a_1)\left(1-\frac{x_4}{a_1}\right)\left(1-\frac{x_1x_4}{a_2}\right)} \\
 & + \frac{1}{(1-a_2x_1)\left(1-\frac{x_2}{a_2x_3}\right)\left(1-\frac{x_3}{a_2}\right)\left(1-\frac{a_2x_3}{a_1}\right)(1-a_1a_2x_3)\left(1-\frac{x_4}{a_2x_3}\right)}.
 \end{aligned}$$

As for \mathbb{C}^3/Z_3 and \mathbb{F}_0 , also the $N = 1$ generating function for dP_1 can be written in a form that recalls a localization formula for the $N = 1$ field theory moduli space, which is a six dimensional variety acted by a total of six flavor and baryonic symmetries. It would be interesting to investigate the general properties of the $N = 1$ moduli space varieties and to see in particular whether these varieties are toric and the previous formulae can be interpreted as a localization.

5.5.5 Generating functions for $N > 1$

The generating functions for $N > 1$ can be obtained as usual by plethystic exponentiation and resummation over the points of the considered decomposition, according to the general formula (2.2). Using the GKZ decomposition, with the multiplicities already obtained for the $N = 1$ case, the generating functions for general N are given by:

$$\sum_{N=0}^{\infty} g_N(\{x_i\})\nu^N = \sum_{\beta, \beta' \in R1} m_1(\beta, \beta') \text{PE}_{\nu}[g_{1, \beta, \beta'}^{R1}(x_i)] + \sum_{\beta, \beta' \in R2} m_2(\beta, \beta') \text{PE}_{\nu}[g_{1, \beta, \beta'}^{R2}(x_i)] \quad (5.61)$$

For the case $N = 2$ we compute:

$$g_2(t, b) = \frac{F(t, b)}{(b^6 - 1)(b^4 - t^2)^4(b^2t^2 - 1)^5(t^3 - 1)^3(b^6t^3 - 1)} \quad (5.62)$$

$$\begin{aligned}
 F(t, b) = & b^8(t^{14}(-t^6 + 6t^3 + 3)b^{22} + t^{12}(4t^6 - 21t^3 - 15)b^{20} + t^{10}(-2t^{12} - 6t^9 + 5t^6 + 3t^3 + 32)b^{18} \\
 & - t^5(3t^{15} - 61t^{12} + 39t^9 - 24t^6 + 16t^3 + 3)b^{16} - t^6(9t^{15} - 30t^{12} + 85t^9 + 7t^6 - 30t^3 + 7)b^{14} \\
 & + (-7t^{22} - 6t^{19} + 25t^{16} - 32t^{13} + 67t^{10} - 48t^7 + t^4)b^{12} \\
 & + t^2(-5t^{18} + 64t^{15} - 91t^{12} + 48t^9 - 29t^6 + 6t^3 + 7)b^{10} \\
 & + (-t^{21} + 14t^{18} - 53t^{15} + 44t^{12} + 56t^9 - 19t^6 + 6t^3 + 1)b^8 \\
 & + t^4(3t^{15} + 15t^{12} - 17t^9 + 24t^6 - 48t^3 - 1)b^6 + t^2(4t^{15} - 44t^{12} + 9t^9 - 9t^6 + 6t^3 + 2)b^4 \\
 & + t^9(-4t^6 + 27t^3 + 9)b^2 + t^7(t^6 - 6t^3 - 3)) \quad (5.63)
 \end{aligned}$$

We explicitly checked for the considered cases that the dimension of the moduli space, equal to the order of the pole for the generating function when t and b approach 1, is $3N + G - 1$ as expected.

The generators of the moduli space and their relations can be studied by computing the plethystic logarithm of the generating functions. In the $N = 1$ case we checked that the plethystic logarithm correctly reproduces the 10 chiral fields generators together with their 9 F-term relations (recall that we get only one relation (5.49) from the two (5.48)).

The case $N = 2$ for dP_1 is more interesting. The first terms in the expansion of the plethystic logarithm of $g_2(t, b)$ in (5.62) are:

$$b^6 + \left(\frac{6}{b^4} + 12b^2 \right) t^2 + (9 + b^6) t^3 - \dots \tag{5.64}$$

The first positive terms can be matched with 29 generators of the moduli space for $N = 2$: in fact at level t^3 we find the 9 known mesons of dP_1 (this matches with the number of generators over the integers for the dual fan), and 20 baryonic generators. The baryonic generators can be identified as follows:

$$\begin{aligned} b^6 &\rightarrow (\mathbf{Z}, \mathbf{Z}) & t^3 b^6 &\rightarrow (\mathbf{Y}_1 \mathbf{Y}_2 \mathbf{Y}_3, \mathbf{Z}) \\ \frac{6t^2}{b^4} &\rightarrow (\mathbf{U}_1, \mathbf{U}_1), (\mathbf{U}_1, \mathbf{U}_2), (\mathbf{U}_2, \mathbf{U}_2), (\tilde{\mathbf{U}}_1, \tilde{\mathbf{U}}_1), (\tilde{\mathbf{U}}_1, \tilde{\mathbf{U}}_2), (\tilde{\mathbf{U}}_2, \tilde{\mathbf{U}}_2) \\ 12b^2 t^2 &\rightarrow (\mathbf{Y}_1, \mathbf{Y}_1), (\mathbf{Y}_2, \mathbf{Y}_2), (\mathbf{Y}_3, \mathbf{Y}_3), (\mathbf{V}_1, \mathbf{V}_1), (\mathbf{V}_1, \mathbf{V}_2), (\mathbf{V}_2, \mathbf{V}_2) \\ &\rightarrow (\mathbf{Y}_2, \mathbf{V}_1), (\mathbf{Y}_2, \mathbf{V}_2), (\mathbf{ZU}_1, \mathbf{Y}_1), (\mathbf{ZU}_2, \mathbf{Y}_1), (\tilde{\mathbf{U}}_1 \mathbf{Z}, \mathbf{Y}_3), (\tilde{\mathbf{U}}_2 \mathbf{Z}, \mathbf{Y}_3) \end{aligned}$$

where (\mathbf{X}, \mathbf{Y}) stands for the $N = 2$ color indices contraction: $\epsilon_{i,j} \epsilon^{a,b} \mathbf{X}_a^i \mathbf{Y}_b^j$.

6. The Molien formula: checks for $N > 1$

The baryonic generating functions found in the previous sections can be checked against an explicit field theory computation, at least for small values of N . The problem of finding invariants under the action of a continuous group is the hearth of invariant theory and goes back to the nineteenth century, as most of the concepts necessary for its solution, like syzygies and free resolutions, all amenable to Hilbert. Modern advances, such as the discovery of Groebner basis, gave an algorithmic way of solving such problems and the advent of computer algebra programs made some computations really doable.

The generating functions for a fixed number of colors N can be reduced to a problem for polynomial rings as follows. Consider an $\mathcal{N} = 1$ supersymmetric gauge theory with F elementary fields X and a gauge group \mathcal{G} . Since we are discussing the chiral ring we can replace \mathcal{G} with its complexification \mathcal{G}_c . For quiver theories, the elementary fields consist of $N \times N$ matrices. Consider a polynomial ring in FN^2 variables $\mathbb{C}[X_{ij}]$ made with the entries of these matrices. The F-terms give matrix relations whose entries are polynomial equations. We can collect all the polynomial F-term equations in an ideal \mathcal{I} and define the quotient ring

$$\mathcal{R}[X_{ij}] = \mathbb{C}[X_{ij}] / \mathcal{I} \tag{6.1}$$

The gauge group \mathcal{G}_c and the global symmetry group act naturally on the ring and we can grade the elements of \mathcal{R} with gauge and global charges. Denoting with t_i the global Abelian charges and with z_i the charges under the Cartan subgroup of the gauge group \mathcal{G}_c , we can write the generating function, or Hilbert series, of the graded ring \mathcal{R} ,

$$H_{\mathcal{R}}(t; z) = \sum_{nm} a_{nm} z^n t^m \tag{6.2}$$

which can be arranged to be a power series in the global charges t and a Laurent expansion in the gauge charges z . The full gauge group \mathcal{G}_c acts on the quotient ring \mathcal{R} and, since the gauge symmetry commutes with the global symmetry, all the elements of \mathcal{R} with given charge t^m form a (not necessarily irreducible) representation of \mathcal{G}_c . Therefore, the coefficient of t^m in equation (6.2) is the character of a \mathcal{G}_c representation,

$$H_{\mathcal{R}}(t; z) = \sum_{m=0}^{\infty} \chi^m(z) t^m = \sum_{m=0}^{\infty} \left(\sum_i a_i^m \chi^{(i)}(z) \right) t^m \tag{6.3}$$

Here we have denoted with $\chi^{(i)}$ the irreducible representations of \mathcal{G}_c and decomposed the representation on the elements of charge t^m into irreducible ones. The generating function for invariants is given by the projection onto the trivial representation with character $\chi^{(0)} = 1$,

$$H_{\mathcal{R}}^{\text{inv}}(t) = \sum_{m=0}^{\infty} a_0^m t^m \tag{6.4}$$

The projection can be easily done by averaging $H(t; z)$ on the gauge group with the Weyl measure. The latter has indeed the property to keep only the contribution of the trivial representation

$$\int d\mu(z) \chi^{(i)}(z) = \delta_{i,0} \tag{6.5}$$

For a given group G with rank r we can explicitly write the measure as a multi-contour integral

$$\frac{1}{|W|} \prod_{j=1}^r \int_{|z_j|=1} \frac{dz_j}{2\pi i z_j} \left(1 - z^{h(\alpha)} \right) \tag{6.6}$$

where $h(\alpha)$ are the weights of the adjoint representation and $|W|$ is the order of the Weyl group. We finally get the Molien formula:

$$H_{\mathcal{R}}^{\text{inv}}(t) = \frac{1}{|W|} \prod_{j=1}^r \int_{|z_j|=1} \frac{dz_j}{2\pi i z_j} \left(1 - z^{h(\alpha)} \right) H_{\mathcal{R}}(t; z) \tag{6.7}$$

Since the multi-contour integrals can be evaluated with the residue theorem, the real problem in using equation (6.7) is the determination of the integrand, that is the Hilbert series of the quotient ring \mathcal{R} . Fortunately, this is the kind of problems that modern commutative algebra made algorithmic and that can be easily solved with computer algebra programs. For example Macaulay2 naturally deals with polynomial rings and it has a build-in command *hilbertSeries*. For moderate values of FN^2 , the computation takes fractions

of second, but it can become too hard with a standard computer already at $N = 3$ and a number of fields F of order 10. In these cases, one can still truncate the computation at a maximum degree in t and get a sensible result.

It is worth mentioning that special care has to be taken when the moduli space is not irreducible and at certain points in moduli space new branches are opening up. This is the case for example for $\mathcal{N} = 2$ theories that have additional Coulomb branches and was treated in [10]. The general case that we want to address is the following. Suppose that we have an $N = 1$ F-term equation where one of the elementary fields can be factorized: $X_0 F(X) = 0$ where $F(X)$ is a polynomial not containing X_0 . Considering the $N = 1$ moduli space as a fibration over the line parametrized by X_0 , we see that the dimension of the fiber increases by one unit over $X_0 = 0$: indeed, for $X_0 \neq 0$ we can impose the further constraint $F(X) = 0$ which reduces by one the dimension of the fiber. This means that a new branch opens up at X_0 and the full moduli space is reducible. This is the case for \mathbb{F}_0 and dP_1 as discussed above in detail. We may want to determine the generating function for a given irreducible component of the moduli space, in particular the closure of the open set $X_0 \neq 0$, or, for $N > 1$, of $\det(X_0) \neq 0$. This is done with a standard trick. Add a new element q to the ring and a new equation, $q \det(X_0) - 1$ to the ideal \mathcal{I} . Clearly, the new equation prevents $\det(X_0)$ from being zero. The irreducible component of the moduli space we are interested in is obtained by projecting the variety defined by the new ideal $\tilde{\mathcal{I}} = (\mathcal{I}, q \det(X_0) - 1)$ on the space parameterized by the X, X_0 and taking the closure. This can be done by eliminating q from the ideal $\tilde{\mathcal{I}}$. This defines the elimination ideal \mathcal{J} that can be computed with the Macaulay2 command *eliminate*. If we define

$$\mathcal{R}[X_{ij}] = \mathbb{C}[X_{ij}]/\mathcal{J} \tag{6.8}$$

we can now proceed as before, compute the Hilbert series of this ring and project it onto gauge invariants with the Molien formula. This would give us the generating function for the particular irreducible component of the moduli space.

We now present some explicit examples based on the conifold and \mathbb{F}_0 . The other cases presented in this paper can be checked similarly, at least for small values of number of fields, F , and number of D-branes, N . When N increases it is necessary to truncate the series to a maximum degree.

6.1 Example: $N = 2$ for the conifold

The generating function of the conifold for $N = 2$ was explicitly computed in [12] and is given in equation (3.4). For $N = 2$ we have four fields $\mathbf{A}_i, \mathbf{B}_i$ that are two-by-two matrices, whose entries we denote by $a_i^{p,q}, b_i^{p,q}$. The four matrix F-term equations

$$\mathbf{A}_1 \mathbf{B}_i \mathbf{A}_2 = \mathbf{A}_2 \mathbf{B}_i \mathbf{A}_1, \quad \mathbf{B}_1 \mathbf{A}_i \mathbf{B}_2 = \mathbf{B}_2 \mathbf{A}_i \mathbf{B}_1$$

give rises to sixteen polynomial equations for the $a_i^{p,q}, b_i^{p,q}$ which generates an ideal \mathcal{I} in the polynomial ring $\mathbb{C}[a_i^{p,q}, b_i^{p,q}]$. The element (g, \bar{g}) of the complexified gauge group $SL(2) \times SL(2)$ acts on the matrices as $\mathbf{A}_i \rightarrow g \mathbf{A}_i \bar{g}^{-1}$ and $\mathbf{B}_i \rightarrow \bar{g} \mathbf{B}_i g^{-1}$. All the entries $a_i^{p,q}$

and $b_i^{p,q}$ transform with a definite charge under the Cartan subgroup

$$g = \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}$$

$$\bar{g} = \begin{pmatrix} w & 0 \\ 0 & 1/w \end{pmatrix} \tag{6.9}$$

We further assign chemical potential t_1 to the eight fields $a_i^{p,q}$ and chemical potential t_2 to the eight fields $b_i^{p,q}$. The sixteen F-term constraints generating the ideal \mathcal{I} transform homogeneously under the gauge and global charges. We can thus grade the quotient ring

$$\mathcal{R}_{N=2} = \mathbb{C}[a_i^{p,q}, b_i^{p,q}]/\mathcal{I} \tag{6.10}$$

with four charges, corresponding to chemical potentials, two gauge z, w and two global t_1, t_2 . The Hilbert series of $\mathcal{R}_{N=2}$ can be computed using Macaulay2

$$H_{\mathcal{R}}(t_1, t_2; z, w) = \frac{P(t_1, t_2; z, w)}{(1-t_1zw)(1-t_1\frac{z}{w})(1-t_1\frac{w}{z})(1-\frac{t_1}{zw})(1-t_2zw)(1-t_2\frac{z}{w})(1-t_2\frac{w}{z})(1-\frac{t_2}{zw})}$$

$$P(t_1, t_2; z, w) = 1 + 4t_1^3t_2 + 4t_1t_2^3 + 6t_1^2t_2^2 + t_1^4t_2^4 - 2(t_2t_1^2 + t_1t_2^2 + t_2^2t_1^3 + t_1^2t_2^3) \left(w + \frac{1}{w}\right) \left(z + \frac{1}{z}\right)$$

$$+ t_1^2t_2^2 \left(w + \frac{1}{w}\right)^2 \left(z + \frac{1}{z}\right)^2 \tag{6.11}$$

The Molien formula now reads

$$g_2(t_1, t_2; \mathcal{C}) = \int_{|w|=1} \frac{dw(1-w^2)}{2\pi iw} \int_{|z|=1} \frac{dz(1-z^2)}{2\pi iz} H_{\mathcal{R}}(t_1, t_2; z, w) \tag{6.12}$$

Some attention should be paid in performing the contour integrals. Recall that $H_{\mathcal{R}}$ gives the generating function for the ring \mathcal{R} when expanded in power series in t_1 and in t_2 which are supposed to be complex numbers of modulus less than one. This should be taken into account when performing the contour integrals on the unit circles $|z| = |w| = 1$. For example, the first contour integration in z takes contribution only from the residues in the points $t_1w, t_1/w, t_2w, t_2/w$ lying inside the unit circle $|z| = 1$ (we take $|t_i| < 1, |w| = 1$). Similar arguments apply to the second integration. After performing the two integrals we obtain

$$g_2(t_1, t_2) = \frac{1 + t_1t_2 + t_1^2t_2^2 - 3t_1^4t_2^2 - 3t_1^2t_2^4 + t_1^5t_2^3 + t_1^3t_2^5 - 3t_1^3t_2^3 + 4t_1^4t_2^4}{(1-t_1^2)^3(1-t_1t_2)^3(1-t_2^2)^3}, \tag{6.13}$$

which perfectly coincides with equation (3.4).

6.2 Example: $N = 1$ and $N = 2$ for $\frac{3}{4}\mathbb{F}_0$ - reducibility of the moduli space

We now consider an example where the moduli space is not irreducible. We consider the $\frac{3}{4}\mathbb{F}_0$ case in order to limit the number of equations involved. The following discussion applies to \mathbb{F}_0 and dP_1 as well. For $N = 1$ we consider the polynomial ring

$$\mathcal{R} = \mathbb{C}[\mathbf{A}_i, \mathbf{B}_1, \mathbf{C}_i, \mathbf{D}_1]/\mathcal{I} \tag{6.14}$$

with six variables. There are two F-term equations

$$\mathcal{I} = (\mathbf{A}_1 \mathbf{B}_1 \mathbf{C}_2 = \mathbf{A}_2 \mathbf{B}_1 \mathbf{C}_1, \mathbf{C}_1 \mathbf{D}_1 \mathbf{A}_2 = \mathbf{C}_2 \mathbf{D}_1 \mathbf{A}_1)$$

The Hilbert series for this polynomial ring is

$$H_{\mathcal{R}}(t_1, t_2) = \frac{1 - 2t_1^2 t_2 + t_1^2 t_2^2}{(1 - t_1)^4 (1 - t_2)^2}. \quad (6.15)$$

As already discussed, the variety defined by \mathcal{I} is not irreducible: we are interested in the closure of the open set $\mathbf{B}_1, \mathbf{D}_1 \neq 0$. We then define a new ideal by adding two new variables q_1, q_2 to \mathcal{R} ,

$$\tilde{\mathcal{R}} = \mathbb{C}[\mathbf{A}_i, \mathbf{B}_1, \mathbf{C}_i, \mathbf{D}_1, q_1, q_2] / \tilde{\mathcal{I}} \quad (6.16)$$

and two new generators to the ideal \mathcal{I}

$$\tilde{\mathcal{I}} = (\mathcal{I}, q_1 \mathbf{B}_1 - 1, q_2 \mathbf{D}_1 - 1)$$

The closure of the open set $\mathbf{B}_1, \mathbf{D}_1 \neq 0$ is obtained by eliminating q_1 and q_2 . This can be done in a polynomial way by using the Groebner basis and the algorithm is implemented in Macaulay2 in the command *eliminate*. In our case the elimination ideal is just

$$\mathcal{J} = (\mathbf{A}_1 \mathbf{C}_2 - \mathbf{A}_2 \mathbf{C}_1)$$

and the Hilbert series of

$$\mathcal{R}' = \mathbb{C}[\mathbf{A}_i, \mathbf{B}_1, \mathbf{C}_i, \mathbf{D}_1] / \mathcal{J} \quad (6.17)$$

is

$$H_{\mathcal{R}'}(t_1, t_2) = \frac{1 + t_1}{(1 - t_1)^3 (1 - t_2)^2}$$

which indeed coincides with the $g_1(t_i, \frac{3}{4}\mathbb{F}_0)$ generating function given in equation (5.46). The $N = 2$ generating function should be computed in a similar way. The fields $\mathbf{A}_i, \mathbf{B}_1, \mathbf{C}_i, \mathbf{D}_1$ are now two-by-two matrices, for a total of 24 independent entries. The ideal \mathcal{I} now contains 8 polynomial equations, given by

$$\tilde{\mathcal{I}} = (\mathcal{I}, q_1 \det \mathbf{B}_1 - 1, q_2 \det \mathbf{D}_1 - 1),$$

and the elimination ideal \mathcal{J} is obtained by *eliminating* q_1 and q_2 . The Hilbert series of \mathcal{J} graded with the gauge charges is a rational function $H_{\mathcal{R}}(t_1, t_2; z_1, z_2, z_3, z_4)$ whose expression is too long to be reported here. Four integrations using the residue theorem finally give the $N = 2$ generating function given in (5.46).

7. Conclusions

In this paper we have performed a further step in the understanding and the computation of the generating functions for the chiral ring of the superconformal gauge theories living on branes at CY singularities. We have reinforced the conjecture that the generating functions for N colors can be computed simply in terms of the $N = 1$ generating functions

through the plethystic program. We tested this conjecture, suggested by a computation with D3-branes in the dual background, in field theory for small values of N . It would be interesting to perform checks for large N as well as to investigate the statistical properties of the resulting generating functions.

In particular we have made an explicit investigation of the properties of the complete $N = 1$ generating function and we have compared the result with a geometrical computation. The emerging structure reveals once more the deep interplay between the quiver gauge theory and the algebraic geometry of the CY. In particular we found an intriguing relation between the decomposition of the $N = 1$ generating function in sector of given baryonic charge and the discretized Kähler moduli space of the CY.

The geometrical structure of the complete moduli space for N colors, which is obtained by the N -fold symmetrized product of the CY by adding the baryonic directions is still poorly understood. We have seen that already for $N = 1$ the moduli space is rich and interesting. We leave for future work the understanding of the geometric structure of these moduli spaces.

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A. Singular $\mathcal{N} = 2$ horizons

In this appendix we discuss the $\mathcal{N} = 2$ supersymmetric theories obtained at orbifold singularities of the form $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}$ and we make some general observations about the corresponding generating functions. Since the factor \mathbb{C} in the geometry factorizes, we can immediately write the following expression,

$$g_1\left(\{t_i\}, t; \frac{\mathbb{C}^2}{\mathbb{Z}_n} \times \mathbb{C}\right) = \frac{1}{1-t} \cdot g_1(\{t_i\}; \mathbb{C}^2/\mathbb{Z}_n) \tag{A.1}$$

where t is a chemical potential for the \mathbb{C} factor and t_i a set of chemical potential for the four dimensional singularity $\mathbb{C}^2/\mathbb{Z}_n$. We count BPS operators on the Higgs branch of the theory without including mixed Higgs-Coulomb branches.

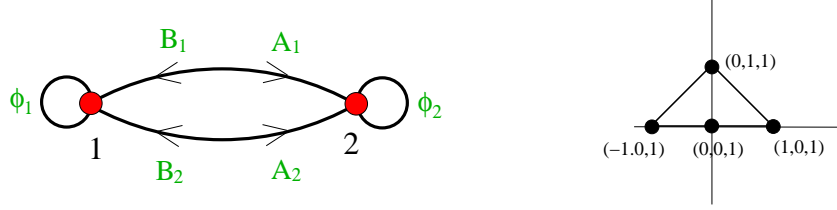


Figure 17: Quiver and toric diagram for $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$.

A.1 $\mathbb{C}^2/\mathbb{Z}_2$

The case of the CY singularity $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ was considered in detail in [12]. The theory has $\mathcal{N} = 2$ supersymmetry with two vector multiplets in the adjoint representation of $SU(N)$ and two bi-fundamental hypermultiplets. In $\mathcal{N} = 1$ notation we have six chiral multiplets denoted as $\phi_1, \phi_2, \mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$, with a superpotential

$$W = \phi_1(\mathbf{A}_1\mathbf{B}_1 - \mathbf{A}_2\mathbf{B}_2) + \phi_2(\mathbf{B}_2\mathbf{A}_2 - \mathbf{B}_1\mathbf{A}_1) \quad (\text{A.2})$$

Let us consider the $N = 1$ generating function, which is given by

$$g_1\left(t_1, t_2, x; \frac{\mathbb{C}^2}{\mathbb{Z}_2}\right) = \frac{1 - t_1 t_2}{(1 - t_1 x) \left(1 - \frac{t_1}{x}\right) (1 - t_2 x) \left(1 - \frac{t_2}{x}\right)} \quad (\text{A.3})$$

This function describes a three dimensional moduli space which is a complete intersection, generated by four fields satisfying one F-term relation $\mathbf{A}_1\mathbf{B}_1 = \mathbf{A}_2\mathbf{B}_2$. We assign chemical potential $t_1 = tb$ to \mathbf{A}_i and $t_2 = t/b$ to \mathbf{B}_i where b measures the baryonic charge. The parameter x counts the difference between indices 1 and 2. Let us introduce two more chemical potentials, $q_1 = tx$ and $q_2 = t/x$. These potentials are natural conjugate variables for the two coordinates of \mathbb{C}^2 . They count the number of fields which descend by the orbifold action from the first (second) adjoint field, respectively.

Similarly to the conifold case, we can expand the $N = 1$ generating function in baryonic charges

$$g_1\left(t_1, t_2; \frac{\mathbb{C}^2}{\mathbb{Z}_2}\right) = \sum_{B=-\infty}^{\infty} b^B g_{1,B}\left(t; \frac{\mathbb{C}^2}{\mathbb{Z}_2}\right)$$

$$g_{1,B}\left(t, x; \frac{\mathbb{C}^2}{\mathbb{Z}_2}\right) = \frac{t^{|B|} x^{|B|}}{\left(1 - \frac{1}{x^2}\right) (1 - t^2 x^2)} + \frac{t^{|B|} x^{-|B|}}{(1 - x^2) \left(1 - \frac{t^2}{x^2}\right)} \quad (\text{A.4})$$

The same result can be obtained by localization and by using the auxiliary GKK partition function which reproduces two one-dimensional cones corresponding to the two Weyl chambers of $SU(2)$.

A.1.1 $\mathbb{C}^2/\mathbb{Z}_2$ as sum over Young tableaux

The baryonic generating functions for $\mathbb{C}^2/\mathbb{Z}_2$ can be given an interpretation as sum over Young tableaux.

There is indeed an intriguing relation between the BPS partition functions for CY of the form $X \times \mathbb{C}$ and Nekrasov's partition function for $\mathcal{N} = 2$ U(1) gauge theories defined on the surface X . The relation is defined by the following identity [72–74],

$$\sum_{N=1}^{\infty} \nu^N \text{ChH}^0(S^N(X), \mathcal{O}) = \exp\left(\sum_{k=1}^{\infty} \frac{\nu^k g_{1,0}(t_i^k; X)}{k}\right) = \prod_I Z(q_1^I, q_2^I, \nu; \mathbb{C}^2) \quad (\text{A.5})$$

In this identity, $g_{1,0}(t_i)$ is the mesonic $N = 1$ generating function which is also the partition function of holomorphic functions on X . The first equality in equation (A.5) is precisely the statement that the mesonic BPS operators count holomorphic functions on the N -fold symmetric product of X . The last equality follows from the computation of $g_{1,0}$ in terms of localization

$$g_{1,0} = \sum_I \frac{1}{(1 - q_1^I)(1 - q_2^I)} \quad (\text{A.6})$$

Here I labels the fixed points and (q_1^I, q_2^I) are the weights for the T^2 action in a smooth resolution of X . Since the $g_{1,0}$ partition function decomposes as the sum over elementary partition functions for copies of \mathbb{C}^2 , the last equality follows.

Equation (A.5) can be reinterpreted as the K-theory version of Nekrasov's U(1) partition function for the case of a surface X [74] and written in terms of Young tableaux. Indeed the partition function for \mathbb{C}^2 $Z(q_1, q_2, \nu; \mathbb{C}^2)$ can be written as a sum over Young tableaux

$$Z(q_1, q_2, \nu; \mathbb{C}^2) = \sum_Y \frac{\nu^{|Y|}}{\prod_{s \in Y} (1 - q_1^{-l(s)} q_2^{a(s)+1}) (1 - q_1^{1+l(s)}) (1 - q_2^{-a(s)})} \quad (\text{A.7})$$

There is a similar result for baryonic partition functions. Since we can write

$$g_{1,B} = \sum_I \frac{q_0^I}{(1 - q_1^I)(1 - q_2^I)} \quad (\text{A.8})$$

we have the following expression,

$$\exp\left(\sum_{r=1}^{\infty} \frac{\nu^r g_{1,B}(t_i^r; X)}{r}\right) = \sum_{N=1}^{\infty} \nu^N \text{ChH}^0(S^N(X), \mathcal{O}(B)) = \prod_I Z(q_1^I, q_2^I, \nu q_0^I; \mathbb{C}^2). \quad (\text{A.9})$$

This expresses the generating function as an expansion in products of series over Young tableaux. For the case of orbifolds, we expect that this simplifies to a sum over a single set of tableaux as in [74].

The identity seems to be related to the blow-up formula of Nakajima [72] thus reinforcing the relation of the baryonic charge B with the Kähler modulus.

A.1.2 An expression for g_N for $\mathbb{C}^2/\mathbb{Z}_2$

Here we give a computation inspired by the previous discussion that may help in simplifying the higher N generating functions.

The generating function for 1 D-brane and baryonic charge B was computed to be

$$g_{1,B}(q_1, q_2) = \frac{q_1^{|B|}}{(1 - q_1^2) \left(1 - \frac{q_2}{q_1}\right)} + \frac{q_2^{|B|}}{\left(1 - \frac{q_1}{q_2}\right) (1 - q_2^2)} \quad (\text{A.10})$$

We can now take the Plethystic Exponential for this expression, keeping track of the baryon number. The simplest way of doing this is to take $\text{PE}[b^B g_{1,B}]$:

$$\text{PE}[b^B g_{1,B}] = \exp\left(\sum_{k=1}^{\infty} \frac{\nu^k b^{kB} q_1^{k|B|}}{k(1 - q_1^{2k}) \left(1 - \frac{q_2^k}{q_1^k}\right)}\right) \exp\left(\sum_{k=1}^{\infty} \frac{\nu^k b^{kB} q_2^{k|B|}}{k \left(1 - \frac{q_1^k}{q_2^k}\right) (1 - q_2^{2k})}\right) \quad (\text{A.11})$$

This form of the equation can be compared with the generating function for the two dimensional complex plane, \mathbb{C}^2 ,

$$g(\nu; t_1, t_2; \mathbb{C}^2) = \exp\left(\sum_{k=1}^{\infty} \frac{\nu^k}{k(1 - t_1^k)(1 - t_2^k)}\right) = \sum_{N=0}^{\infty} \nu^N g_N(t_1, t_2; \mathbb{C}^2), \quad (\text{A.12})$$

to write an expression

$$\text{PE}[b^B g_{1,B}] = g\left(\nu b^B q_1^{|B|}; q_1^2, \frac{q_2}{q_1}; \mathbb{C}^2\right) \cdot g\left(\nu b^B q_2^{|B|}; \frac{q_1}{q_2}, q_2^2; \mathbb{C}^2\right), \quad (\text{A.13})$$

precisely as explained above. We can now use the expansion in the number of branes to demonstrate that the baryon number dependence is simple,

$$\text{PE}[b^B g_{1,B}] = \sum_{N_1=0}^{\infty} \nu^{N_1} b^{BN_1} q_1^{|B|N_1} g_{N_1}\left(q_1^2, \frac{q_2}{q_1}; \mathbb{C}^2\right) \sum_{N_2=0}^{\infty} \nu^{N_2} b^{BN_2} q_2^{|B|N_2} g_{N_2}\left(\frac{q_1}{q_2}, q_2^2; \mathbb{C}^2\right), \quad (\text{A.14})$$

In order to perform the sum over all baryon numbers, we use the identity

$$\sum_{B=-\infty}^{\infty} b^B t^{|B|} = \frac{1 - t^2}{(1 - tb) \left(1 - \frac{t}{b}\right)} \quad (\text{A.15})$$

and get

$$\begin{aligned} \sum_{B=-\infty}^{\infty} \text{PE}[b^B g_{1,B}] &= \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \nu^{N_1+N_2} \frac{1 - q_1^{2N_1} q_2^{2N_2}}{[1 - (q_1 b)^{N_1} (q_2 b)^{N_2}] [1 - (\frac{q_1}{b})^{N_1} (\frac{q_2}{b})^{N_2}]} \times \\ &\quad \times g_{N_1}\left(q_1^2, \frac{q_2}{q_1}; \mathbb{C}^2\right) g_{N_2}\left(\frac{q_1}{q_2}, q_2^2; \mathbb{C}^2\right). \end{aligned} \quad (\text{A.16})$$

From this expression we can easily extract the generating function for a fixed N number of D-branes,

$$\begin{aligned} g_N\left(q_1, q_2, b; \frac{\mathbb{C}^2}{\mathbb{Z}_2}\right) &= \sum_{N_1+N_2=N} \frac{1 - q_1^{2N_1} q_2^{2N_2}}{[1 - (q_1 b)^{N_1} (q_2 b)^{N_2}] [1 - (\frac{q_1}{b})^{N_1} (\frac{q_2}{b})^{N_2}]} \times \\ &\quad \times g_{N_1}\left(q_1^2, \frac{q_2}{q_1}; \mathbb{C}^2\right) g_{N_2}\left(\frac{q_1}{q_2}, q_2^2; \mathbb{C}^2\right). \end{aligned} \quad (\text{A.17})$$

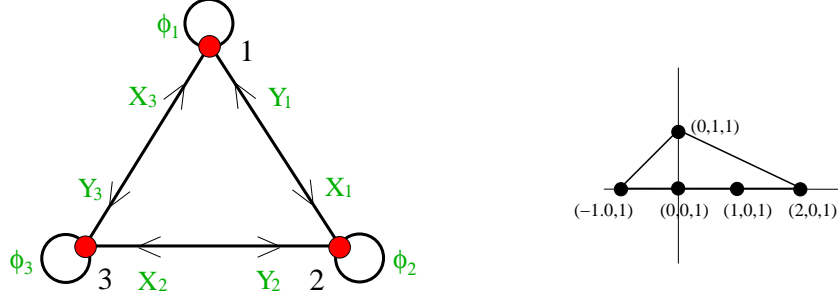


Figure 18: Quiver and toric diagram for $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$.

By determining the first few terms for \mathbb{C}^2 in the expansion (A.12),

$$\begin{aligned}
 g_1(t_1, t_2; \mathbb{C}^2) &= \frac{1}{(1-t_1)(1-t_2)} \\
 g_2(t_1, t_2; \mathbb{C}^2) &= \frac{1+t_1t_2}{(1-t_1)(1-t_2)(1-t_1^2)(1-t_2^2)},
 \end{aligned} \tag{A.18}$$

we are able to compute the generating functions for $\mathbb{C}^2/\mathbb{Z}_2$ with the following results. For $N = 1$ we obtain,

$$g_1\left(q_1, q_2, b; \frac{\mathbb{C}^2}{\mathbb{Z}_2}\right) = \frac{1}{(1-q_1b)\left(1-\frac{q_1}{b}\right)\left(1-\frac{q_2}{q_1}\right)} + \frac{1}{(1-q_2b)\left(1-\frac{q_2}{b}\right)\left(1-\frac{q_1}{q_2}\right)}. \tag{A.19}$$

For $N = 2$,

$$\begin{aligned}
 g_2\left(q_1, q_2, b; \frac{\mathbb{C}^2}{\mathbb{Z}_2}\right) &= \frac{1+q_1q_2}{(1-q_1^2b^2)\left(1-\frac{q_1^2}{b^2}\right)(1-q_1^2)\left(1-\frac{q_2}{q_1}\right)\left(1-\frac{q_2^2}{q_1^2}\right)} \\
 &+ \frac{1-q_1^2q_2^2}{(1-q_1q_2b^2)\left(1-\frac{q_1q_2}{b^2}\right)(1-q_1^2)\left(1-\frac{q_2}{q_1}\right)\left(1-\frac{q_1}{q_2}\right)(1-q_2^2)} \\
 &+ \frac{1+q_1q_2}{(1-q_2^2b^2)\left(1-\frac{q_2^2}{b^2}\right)\left(1-\frac{q_1}{q_2}\right)\left(1-\frac{q_1^2}{q_2^2}\right)(1-q_2^2)}.
 \end{aligned} \tag{A.20}$$

A.2 $\mathbb{C}^2/\mathbb{Z}_3$

Let us consider the orbifold geometry $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$. The theory has $\mathcal{N} = 2$ supersymmetry with three vector multiplets in the adjoint representation and three bi-fundamental hypermultiplets. In $\mathcal{N} = 1$ notation, this transforms to nine chiral multiplets denoted by ϕ_i , \mathbf{X}_i , \mathbf{Y}_i with $i = 1, 2, 3$. (The charges are written in table 5.) The superpotential is

$$W = \phi_1(\mathbf{X}_1\mathbf{Y}_1 - \mathbf{Y}_3\mathbf{X}_3) + \phi_2(\mathbf{X}_2\mathbf{Y}_2 - \mathbf{Y}_1\mathbf{X}_1) + \phi_3(\mathbf{X}_3\mathbf{Y}_3 - \mathbf{Y}_2\mathbf{X}_2) \tag{A.21}$$

The generating function for one D-brane is relatively easy to obtain once one makes the observation that the moduli space is a complete intersection. The argument goes as

field	U(1) _X	U(1) _Y	B ₁	B ₂	charges
X ₁	1	0	1	0	t ₁ b ₁
Y ₁	0	1	-1	0	t ₂ /b ₁
X ₂	1	0	-1	1	t ₁ b ₂ /b ₁
Y ₂	0	1	1	-1	t ₂ b ₁ /b ₂
X ₃	1	0	0	-1	t ₁ /b ₂
Y ₃	0	1	0	1	t ₂ b ₂

Table 5: Global charges for the bi-fundamental fields of the quiver gauge theory living on the D-brane probing the $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$ singularity.

follows. Excluding the adjoint fields, there are six fields in the quiver. Besides the two flavor symmetries that are dual to isometries in the \mathbb{C}^2 directions, there are also two baryonic symmetries coming from the two Fayet-Iliopoulos terms which can be introduced for three gauge groups. All together we count four symmetries leading to a four-dimensional moduli space generated by six fields. Since there are only two F-term relations, $\mathbf{X}_1 \mathbf{Y}_1 = \mathbf{X}_2 \mathbf{Y}_2 = \mathbf{X}_3 \mathbf{Y}_3$, this manifold is a complete intersection.

This enables us to immediately write down the generating function,

$$g_1 \left(t_1, t_2; b_1, b_2; \frac{\mathbb{C}^2}{\mathbb{Z}_3} \right) = \frac{(1-t_1 t_2)^2}{(1-t_1 b_1) \left(1-\frac{t_2}{b_1}\right) \left(1-\frac{t_1 b_2}{b_1}\right) \left(1-\frac{t_2 b_1}{b_2}\right) \left(1-\frac{t_1}{b_2}\right) (1-t_2 b_2)} \quad (\text{A.22})$$

where we have set the generator charges in the denominator according to table 5. The numerator is obtained from the charges of the F-terms.

Alternatively, one arrives at the exact same result using localization and the auxiliary GKZ polynomial ring. The latter leads to six two dimensional cones parametrized by B_1 and B_2 . These cones correspond to the Weyl chambers of $SU(3)$ depicted in figure 19. For fixed baryonic charges, localization gives

$$g_{1, B_1, B_2}(t_1, t_2; \mathbb{C}^2/\mathbb{Z}_3) = \frac{t_1^{\tilde{B}_1+2\tilde{B}_2}}{(1-t_1^3) \left(1-\frac{t_2}{t_1}\right)} + \frac{t_1^{\tilde{B}_1} t_2^{\tilde{B}_2}}{\left(1-\frac{t_1^2}{t_2}\right) \left(1-\frac{t_2^2}{t_1}\right)} + \frac{t_2^{2\tilde{B}_1+\tilde{B}_2}}{\left(1-\frac{t_1}{t_2}\right) (1-t_2^3)}$$

where $(\tilde{B}_1, \tilde{B}_2)$ is a lattice point in the fundamental chamber, and can be obtained by Weyl reflection of (B_1, B_2) . This is similar to the $\mathbb{C}^3/\mathbb{Z}_2$ case, where the Weyl reflection boils down to taking the absolute value of the baryonic charge, cf. (A.4).

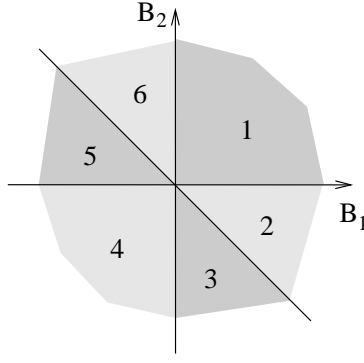


Figure 19: The six cones for $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$.

In detail, we have

$$g_{1,B_1,B_2} \left(t_1, t_2; \frac{\mathbb{C}^2}{\mathbb{Z}_3} \right) = \left\{ \begin{array}{l} \frac{t_1^{B_1+2B_2}}{(1-t_1^3)(1-\frac{t_2}{t_1})} + \frac{t_1^{B_1} t_2^{B_2}}{\left(1-\frac{t_1^2}{t_2}\right)\left(1-\frac{t_2^2}{t_1}\right)} + \frac{t_2^{2B_1+B_2}}{\left(1-\frac{t_1}{t_2}\right)(1-t_2^3)} \quad B_1 \geq 0, B_2 \geq 0 \\ \frac{t_1^{B_1-B_2}}{(1-t_1^3)\left(1-\frac{t_2}{t_1^2}\right)} + \frac{t_1^{B_1+B_2} t_2^{-B_2}}{\left(1-\frac{t_1^2}{t_2}\right)\left(1-\frac{t_2^2}{t_1}\right)} + \frac{t_2^{2B_1+B_2}}{\left(1-\frac{t_1}{t_2}\right)(1-t_2^3)} \quad B_1 \geq 0 \geq B_2 \geq -B_1 \\ \frac{t_1^{B_1-B_2}}{(1-t_1^3)\left(1-\frac{t_2}{t_1^2}\right)} + \frac{t_1^{-B_1-B_2} t_2^{B_1}}{\left(1-\frac{t_1^2}{t_2}\right)\left(1-\frac{t_2^2}{t_1}\right)} + \frac{t_2^{-B_1-2B_2}}{\left(1-\frac{t_1}{t_2}\right)(1-t_2^3)} \quad 0 \leq B_1 \leq -B_2 \\ \frac{t_1^{-2B_1-B_2}}{(1-t_1^3)\left(1-\frac{t_2}{t_1^2}\right)} + \frac{t_1^{-B_2} t_2^{-B_1}}{\left(1-\frac{t_1^2}{t_2}\right)\left(1-\frac{t_2^2}{t_1}\right)} + \frac{t_2^{-B_1-2B_2}}{\left(1-\frac{t_1}{t_2}\right)(1-t_2^3)} \quad B_1 \leq 0, B_2 \leq 0 \\ \frac{t_1^{-2B_1-B_2}}{(1-t_1^3)\left(1-\frac{t_2}{t_1^2}\right)} + \frac{t_1^{B_2} t_2^{-B_1-B_2}}{\left(1-\frac{t_1^2}{t_2}\right)\left(1-\frac{t_2^2}{t_1}\right)} + \frac{t_2^{-B_1+B_2}}{\left(1-\frac{t_1}{t_2}\right)(1-t_2^3)} \quad B_1 \leq 0 \leq B_2 \leq -B_1 \\ \frac{t_1^{2B_1+B_2}}{(1-t_1^3)\left(1-\frac{t_2}{t_1^2}\right)} + \frac{t_1^{-B_1} t_2^{B_1+B_2}}{\left(1-\frac{t_1^2}{t_2}\right)\left(1-\frac{t_2^2}{t_1}\right)} + \frac{t_2^{-B_1+B_2}}{\left(1-\frac{t_1}{t_2}\right)(1-t_2^3)} \quad B_2 \geq -B_1 \geq 0 \end{array} \right.$$

Now g_{1,B_1,B_2} can be summed up for B_1 and B_2 , giving precisely the result of (A.22).

A.3 $\mathbb{C}^2/\mathbb{Z}_n$

For sake of completeness, we derive here the general structure of the $N = 1$ generating function for $\mathbb{C}^2/\mathbb{Z}_n$ by making use of brane tilings [33, 34]. It is easy to see that the polynomial ring is a complete intersection and therefore the $N = 1$ generating function is easy to compute. This could also be computed on a case-by-case basis with Hilbert series in Macaulay2.

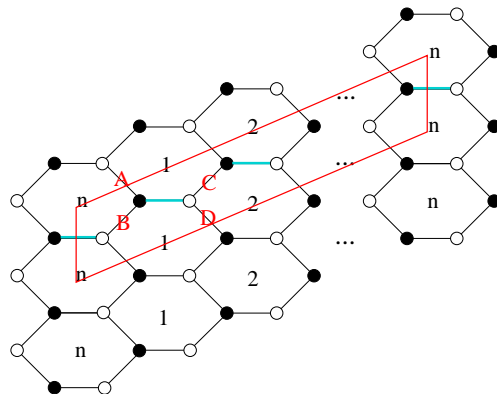


Figure 20: Tiling for $\mathbb{C}^2/\mathbb{Z}_n$. The blue adjoint fields give nontrivial constraints.

The brane tiling for the geometry is shown in figure 20. The edges are in one-to-one correspondence with the $2n$ bifundamental and n adjoint fields in the theory. Black and white vertices give terms in the superpotential. Gauge groups are labeled by the faces in the tiling.

F-terms for the bifundamental fields set the blue adjoint fields equal to each other. On the other hand, F-terms for the adjoint fields give $n - 1$ constraints on the bifundamental fields. For instance,

$$\mathbf{AB} = \mathbf{CD} \tag{A.23}$$

where \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} are indicated in figure 20. The baryonic charges of these equations are zero. Flavor charges are assigned $t_1(t_2)$ for a field that goes southwest (southeast) in figure 20, respectively. For example, \mathbf{A} and \mathbf{D} are assigned a chemical potential t_1 and \mathbf{B} and \mathbf{C} are assigned a chemical potential t_2 . Hence, each F-term carries weight $t_1 t_2$, leading to a $(1 - t_1 t_2)$ factor in the numerator of the generating function. There are $n - 1$ such relations. The general formula for $\mathbb{C}^2/\mathbb{Z}_n$,

$$g_1(t_1, t_2, b_1, \dots, b_{n-1}; \mathbb{C}^2/\mathbb{Z}_n) = \frac{(1 - t_1 t_2)^{n-1}}{\prod_{i=1}^n (1 - t_1 b^{\vec{B}_i}) (1 - t_2 b^{-\vec{B}_i})}$$

where $b^{\vec{B}} \equiv b_1^{B_1} b_2^{B_2} \dots b_{n-1}^{B_{n-1}}$. Here the assignment of baryonic charges in n vectors \vec{B}^i which live in $n - 1$ dimensions can be chosen with any convenient basis which is isomorphic to the simple roots of the A_{n-1} Lie algebra. A possible choice for these n vectors can be a straightforward generalization of table 5, $\vec{B}_1 = (1, 0, 0 \dots 0)$, $\vec{B}_2 = (-1, 1, 0 \dots 0)$, $\vec{B}_3 = (0, -1, 1, 0 \dots 0)$, \dots , $\vec{B}_{n-1} = (0 \dots 0, -1, 1)$, $\vec{B}_n = (0 \dots 0, -1)$.

B. A look at the shiver

In order to understand anomalous baryonic charges, we consider the mirror Calabi-Yau.

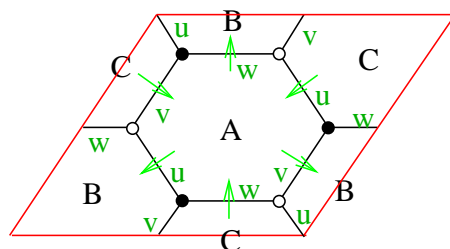


Figure 21: Shiver for $\mathbb{C}^3/\mathbb{Z}_3$.

The geometric description of the mirror [75–77] consists of a double fibration over $W \in \mathbb{C}$,

$$\begin{aligned} W &= P(w, z) \equiv \sum c_{p,q} w^p z^q \\ W &= uv \end{aligned}$$

where $w, z \in \mathbb{C}^*$ and $u, v \in \mathbb{C}$. $P(w, z)$ is the Newton polynomial of the toric diagram and describes a punctured Riemann surface fiber over the W plane. The genus of this surface equals I , i. e. the number of internal points in the toric diagram.

According to the mirror conjecture, the gauge theory arises from D6-branes wrapping three-cycles. These three-cycles intersect over $W = 0$, and open strings at such intersection points give chiral bifundamental matter fields [78]. The three-cycles wrap one-cycles in the Riemann-surface fiber at $W = 0$. They determine a mirror tiling which we will call here the “shiver”. This graph is related to the brane tiling (or dimer graph) by the so-called “antimap”. The detailed description can be found in [37].

The brane tiling lives on a torus and the two nontrivial cycles are related to the flavor charges. The shiver lives on the Riemann surface where the $2I$ nontrivial cycles are related to the anomalous charges. We have the analogy

$$\text{flavor charges : brane tiling} :: \text{anomalous charges : shiver}$$

In the following, we study this analogy through examples.

B.1 Anomalous charges of $\mathbb{C}^3/\mathbb{Z}_3$

Let us consider the shiver of the $\mathbb{C}^3/\mathbb{Z}_3$ theory which is given in figure 21. The edges are in one-to-one correspondence with the arrows in the quiver. The u, v and w represent the three types of fields which differ in their (torsion) D1-charges. It follows that their anomalous U(1) charges are also different,

$$u = a_1 a_2 \quad v = \frac{a_2}{a_1} \quad w = \frac{1}{a_2^2}. \quad (\text{B.1})$$

We see that trivial loops along the green arrows have vanishing charges. The two nontrivial loops have charges $\frac{1}{a_1^2}$ and $\frac{1}{a_1 a_2^3}$, respectively. Therefore, anomalous charges distinguish between the two cycles and in this sense they are similar to the flavor charges in the tiling.

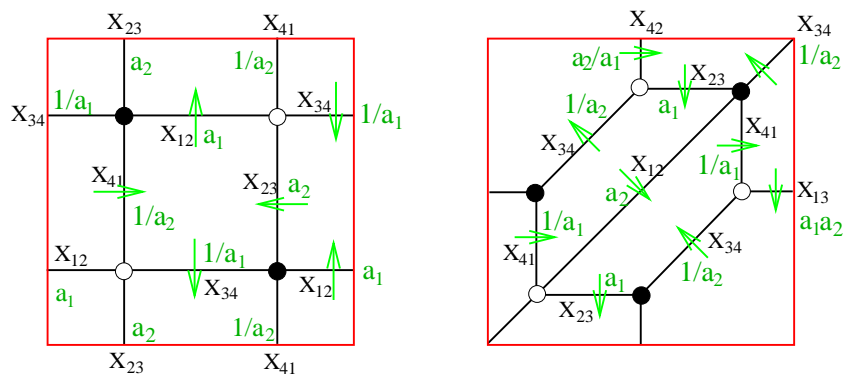


Figure 22: (i) Shiver for \mathbb{F}_0 . (ii) Shiver for dP_1 .

B.2 Anomalous charges of \mathbb{F}_0

The assignment of anomalous charges is shown in figure 7. In the shiver (figure 22), trivial loops along the green arrows again have vanishing charges. The two nontrivial loops have charges a_1^2 and a_2^2 , respectively.

B.3 Anomalous charges of dP_1

The assignment of anomalous charges is shown in figure 11. In the shiver (figure 22), the two nontrivial loops have charges $\frac{a_2^2}{a_1}$ and $\frac{1}{a_1 a_2^2}$, respectively.

C. List of notations

N	number of D-branes
G	number of $SU(N)$ gauge group factors in the theory
F	number of fields (chiral bi-fundamental multiplets) in the theory
t_i	chemical potentials (weights)
x, y, q_i	flavor charge weights
b_i	non-anomalous baryonic weights
a_i	anomalous baryonic weights
R	R-charge
B_i, B'	baryonic charges from the gauge theory
β, β'	baryonic charges from the geometry
A_i	anomalous baryonic charges
$\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$	fields in the gauge theory
\mathcal{R}	polynomial ring
\mathcal{I}, \mathcal{J}	ideals for a polynomial ring
a, b, c, \dots	generators of the auxiliary GKZ ring
$m(B_1, B_2, \dots)$	multiplicities in the GKZ fan

I	number of internal (integral) points in toric diagram
d	number of external points in toric diagram
D_i	toric divisors assigned to the integral points in the toric diagram
x_i	homogeneous coordinates
$g_N(\{t_i\}; CY)$	baryonic generating function for N D-branes probing CY
$PE_\nu[\cdot]$	plethystic exponential with weight ν for the number of D-branes
$Z_{\beta, \beta'}(\{t_i\}; CY)$	partition function from the geometry
$C(\beta)$	hollow polygon in the fiber over the GKZ lattice

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